Math 204B, Winter 2019

Number Theory II, HW 3

Due Wednesday January 30th in class (or by noon).

From Neukirch's book `Algebraic Number Theory`:

- Exercises:

  4 on page 106; 4 on page 115; 1 on page 123

Problem A. Let K be a field extension of $\mathbb{C}$ with an absolute value $|\cdot|$ extending the ordinary one $|\cdot|_\infty$ on the complex numbers. This exercise shows that $K = \mathbb{C}$.

(a) Suppose there exists an $a \in K\backslash\mathbb{C}$. Show that a has a nearest point in $\mathbb{C}$. That is, there exists a $z_0 \in \mathbb{C}$ for which the inequality

$$|a - z| \geq |a - z_0|$$

is valid for all $z \in \mathbb{C}$.

(b) Replacing a by $a - z_0$, and then scaling by a suitable complex number, show the existence of an $a \in K\backslash\mathbb{C}$ satisfying

$$|a - z| \geq |a| > 1$$

for all $z \in \mathbb{C}$.

(c) For an arbitrary $n \in \mathbb{N}$ use $a^n - 1 = \prod_{i=0}^{n-1}(a - \zeta^i)$ to show that $|a - 1| = |a|$.

(d) Deduce that $|a - n| = |a|$ for all $n \in \mathbb{N}$, and conclude $n \leq 2|a|$. (Contradiction.)

Problem B. Let $(K, |\cdot|)$ be a non-discretely valued non-archimedean field of residue characteristic $p = \mathrm{char}(R/\mathfrak{m}) > 0$. Suppose the p-power Frobenius map $R/(p) \longrightarrow R/(p)$ is surjective[1].

(a) Check that the valuation group $|K^\times|$ is generated by the set of all values $|x|$ in the range $|p| < |x| \leq 1$, and deduce that $|K^\times|$ is a p-divisible group.

[1] A complete field K with these properties is called a *perfectoid* field.

(b) Infer from (a) that $\mathfrak{m} = \mathfrak{m}^2$, and conclude that R is not Noetherian (`Hint`: Krull's intersection theorem).