Math 204B, Winter 2019

Number Theory II, HW 4

Due Wednesday February 6th in class (or by noon).

From Neukirch's book `Algebraic Number Theory`:

- Exercises:

  7 on page 115; 1 and 2 on pages 165–166

Problem A. Let K be a field equipped with a non-archimedean absolute value $|\cdot|$ which we assume is non-trivial. We endow the vector space of polynomials $K[t]$ with the norm $\|\cdot\|$ defined as follows.

$$\|f\| = \max\{|a_0|, |a_1|, \dots, |a_n|\} \qquad f = a_0 + a_1 t + \cdots + a_n t^n.$$

(a) Show that $\|\cdot\|$ is multiplicative; meaning that $\forall f, g \in K[t]$ we have

$$\|fg\| = \|f\| \cdot \|g\|.$$

(`Hint`: Adapt the proof of the Gauss Lemma about contents.)

(b) Check that $\|\cdot\|$ extends uniquely to an absolute value on the field $K(t)$ of rational functions.

(c) Is $K(t)$ complete? Prove it or give a divergent Cauchy sequence.

Problem B. Consider the field $F = \mathbb{F}_p(t)$ with the absolute value $|\cdot|_t$ and its completion $\hat{F} = \mathbb{F}_p((t))$; the field of formal Laurent series over $\mathbb{F}_p$.

(a) Argue that F is countable but $\hat{F}$ is uncountable. Deduce that $\hat{F}$ is not an algebraic extension of F.

(b) Choose an element $\gamma \in \hat{F}$ which is transcendental over F and let

$$E = F(\gamma) \qquad K = F(\gamma^p).$$

Observe that the field extension E/K is purely inseparable of degree p.

(c) Show that E and K have the same closures in $\hat{F}$. More precisely that

$$\hat{K} = \hat{E} = \hat{F}.$$

(This example shows that it can happen that a non-trivial field extension collapses upon completion.)