MATH 204B, WINTER 2019

NUMBER THEORY II, HW 5

Due Wednesday February 13th in class (or by noon).

From Neukirch's book `Algebraic Number Theory`:

- Exercises:

  2 on page 152 (take K complete); 1 on page 159; 3 on page 166

Problem A. Let $(K, |\cdot|)$ be a complete non-archimedean field, and suppose E/K is a finite extension with separable residual extension k_E/k_K.

(a) Check that E/K Galois $\Longrightarrow k_E/k_K$ Galois.

(b) Assuming E/K is Galois show that the canonical homomorphism

$$\psi : \mathrm{Gal}(E/K) \longrightarrow \mathrm{Gal}(k_E/k_K)$$

is surjective and $E^{\ker(\psi)}$ is the maximal unramified extension of K in E.

Problem B. Let $p^r > 1$ be a prime power, and let ζ be a primitive p^r-th root of unity in $\bar{\mathbb{Q}}_p$.

(a) Explain why $\mathbb{Q}_p(\zeta)$ is a totally ramified extension of $\mathbb{Q}_p$ of degree $\phi(p^r)$, and the element $1-\zeta$ is a uniformizer of $\mathbb{Q}_p(\zeta)$.

(b) Let $r = 1$. Prove the following identity.

$$\mathbb{Q}_p(\zeta) = \mathbb{Q}_p(\sqrt[p-1]{-p}).$$

(`Hint`: Write $p = u(1-\zeta)^{p-1}$ with a unit $u \equiv -1 \bmod (1-\zeta)$ by Wilson's congruence. Hensel's lemma shows $-u = x^{p-1}$ for some $x \in \mathbb{Q}_p(\zeta)$. Consequently $-p$ is also a $(p-1)$-st power.)