Math 204B, Winter 2019

Number Theory II, HW 6

Due Wednesday February 20th in class (or by noon).

From Neukirch's book `Algebraic Number Theory`:

- Exercises:

  3 on page 159 (assume L/K are local fields); 3 on page 176

Problem A. Here we show $\bar{\mathbb{Q}}_p$ is <u>not</u> complete relative to $|\cdot|_p$.

(a) Let $\mathbb{Q}_{p^{n!}}$ be the unramified extension of $\mathbb{Q}_p$ of degree $n!$. (From class we know that $\mathbb{Q}_{p^{n!}} = \mathbb{Q}_p(\xi_n)$ where $\xi_n \in \bar{\mathbb{Q}}_p$ is a primitive $(p^{n!}-1)$-st root of unity.) Check that $\mathbb{Q}_{p^{n!}} \subset \mathbb{Q}_{p^{(n+1)!}}$ for all n.

(b) Let s_n be the n-th partial sum of the infinite series $\sum_{i=0}^{\infty} \xi_i p^i$. Verify that $s_n \in \mathbb{Q}_{p^{n!}}$, and that the sequence $(s_n)_{n\in\mathbb{N}}$ is Cauchy in $\bar{\mathbb{Q}}_p$.

(c) Suppose $s_n \to \alpha \in C$. Use Krasner's Lemma to see that $\mathbb{Q}_p(s_n) = \mathbb{Q}_p(\alpha)$ for all n sufficiently large. Deduce that $\alpha \in \mathbb{Q}_{p^{n!}}$ for such n.

(d) Fix a large n as in (c) and argue that α has a p-expansion $\alpha = \sum_{i=0}^{\infty} c_i p^i$ in $\mathbb{Q}_{p^{n!}}$ whose coefficients are either 0 or powers of ξ_n.

(e) For $m > n$ compare the two expansions of α modulo p^{m+1} and infer that $\xi_i = c_i$ for all $i \leq m$. (Observing that $\langle \xi_i \rangle \subset \langle \xi_m \rangle$ may be helpful.)

(f) Get the contradiction $\mathbb{Q}_{p^{m!}} = \mathbb{Q}_{p^{n!}}$.

Problem B. In continuation of Problem A we show that the p-adic completion $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ is algebraically closed.

(a) Let $f \in \mathbb{C}_p[X]$ be monic and irreducible. Spell out why $\forall \delta > 0$ there is a monic polynomial $g \in \bar{\mathbb{Q}}_p[X]$ of the same degree such that $\|f - g\| < \delta$.

(b) As explained in class this implies g is irreducible if δ is small enough, and that g moreover has the root exchange property: For any root $\alpha \in \bar{\mathbb{C}}_p$ of f there is a root $\beta \in \bar{\mathbb{C}}_p$ of g such that $\mathbb{C}_p(\alpha) = \mathbb{C}_p(\beta)$.

– conclude that $\alpha \in \mathbb{C}_p$.

Problem C. (Will not be graded.) Let K be a non-archimedean local field with valuation ring R, and normalized[1] absolute value $\|\cdot\|_K$. Let μ be the Haar measure on K with $\mu(R) = 1$. Show that $\mu(xR) = \|x\|_K$ for all $x \in K$.

[1] That is $\|x\|_K = q^{-v_K(x)}$ where q is the size of the residue field.