Math 204B, Winter 2019

Number Theory II, HW 7

Due Wednesday February 27th in class (or by noon).

From Neukirch's book `Algebraic Number Theory`:

- Exercises:

  2, 4, 5 on page 142

Problem A. Here we show that $\mathbb{Q}_p$ has only finitely many extensions of a given degree (in a fixed algebraic closure $\bar{\mathbb{Q}}_p$).

(a) Reduce the question to showing that any finite extension $K/\mathbb{Q}_p$ only has finitely many totally ramified extensions E/K of a given degree n.

(b) As shown in class any such E/K is of the form $E = K(\Pi)$ where $\Pi \in E$ is a uniformizer. Furthermore the minimal polynomial of Π is an Eisenstein polynomial:

$$f(X) = X^n + \pi a_{n-1}X^{n-1} + \cdots + \pi a_1 X + \pi a_0, \qquad n = [E : K].$$

Here $\pi \in K$ is a choice of uniformizer; all $a_i \in R$ and $a_0 \in R^\times$.

– deduce that there is an n-to-one **map** from pairs (E, Π) onto $R^{n-1} \times R^\times$.

(c) Show that the inverse image of $(a_{n-1}, \ldots, a_1, a_0)$ gives rise to the same fields E as the inverse image of any close enough tuple $(b_{n-1}, \ldots, b_1, b_0)$. (`Hint`: Krasner's lemma; or rather a consequence thereof from class.)

(d) Using the compactness of $R^{n-1} \times R^\times$ deduce that there are only finitely many totally ramified E/K of degree n.

Problem B. Let k be any field of characteristic $p > 0$. Here we show that $K = k((t))$ has infinitely many separable extensions of degree p (in a fixed separable closure K^{sep}).

(a) Consider the rational functions $\frac{1}{t^n}$ with $n > 0$ prime-to-p. Suppose $n > n'$ and

$$\frac{1}{t^n} - \frac{1}{t^{n'}} = f^p - f, \qquad f \in K.$$

Argue that $f \notin k[\![t]\!]$ – in other words that $v_K(f) < 0$.

(b) In continuation of (a) check that

$$-n = v_K(f^p - f) = \min\{v_K(f^p), v_K(f)\} = pv_K(f)$$

which contradicts the assumption $p \nmid n$.

(c) Conclude that K has infinitely many p-extensions in K^{sep}. (`Hint`: Use Artin-Schreier theory. By (b) the additive group $K/\wp(K)$ is infinite, where $\wp(f) = f^p - f$ is the Artin-Schreier operator $\wp : K \to K$.)

(d) Assuming k is finite (so that K is a local field) adapt the strategy of Problem A to show that K has only finitely many **tamely** ramified extensions in K^{sep} of any given degree. (`Hint`: Separability of Eisenstein polynomials is what allows you to use Krasner's lemma.)