Math 204B, Winter 2019

Number Theory II, HW 8

Due Wednesday March 6th in class (or by noon).

From Neukirch's book `Algebraic Number Theory`:

- Exercises:

  1 on page 142 (note that $\frac{1}{1-p}$ *should be* $\frac{1}{p-1}$ *here.* ***Hint****: Set* $\log(p) = 0$*)*

Problem A. (This exercise should have been assigned weeks ago.) Let E/K be a finite extension of local fields with uniformizers π and Π. Thus $\pi \sim \Pi^e$ where $e = e(E/K)$ is the ramification index. Let $f = f(E/K)$ be the inertia degree.

(a) Suppose $\beta_1, \dots, \beta_r$ are elements of R_E whose reductions modulo π span $R_E/\pi R_E$ as a k-vector space ($k = R/\pi R$). Show that $\beta_1, \dots, \beta_r$ generate R_E as an R-module. (`Hint`: $R_E = M + \pi R_E$ where $M = R\beta_1 + \cdots + R\beta_r$.)

(b) Suppose $\alpha_1, \dots, \alpha_f$ are elements of R_E whose reductions modulo Π form a k-basis for k_E. Using part (a) show that the set of elements

$$\alpha_i \Pi^j \qquad (i = 1, \dots, f \qquad j = 0, \dots, e-1)$$

form an R-basis for R_E.

(c) Conclude that R_E is a free R-module of rank $[E : K]$.

Problem B. Let E/K be a finite Galois extension of local fields with Galois group $G = \mathrm{Gal}(E/K)$ and higher ramification groups

$$G_i = \{\sigma \in G : v_E\big(\sigma(x) - x\big) > i \ \ \forall x \in R_E\}.$$

Our goal is to show $G_i = \{1\}$ for all sufficiently large i.

(a) Suppose $x_1, \dots, x_r$ generate R_E as an R-module (cf. Problem A). Check that $\sigma \in G$ lies in the ith ramification group G_i if and only if

$$v_E\big(\sigma(x_s) - x_s\big) > i \qquad \forall s = 1, \dots, r.$$

(b) Choose an $N \in \mathbb{N}$ bigger than all finite valuations $v_E\big(\sigma(x_s) - x_s\big)$ where $\sigma \in G$ and $s = 1, \dots, r$ vary. Argue that

$$G_i = \{1\} \qquad \forall i \geq N.$$

(`Hint`: $\sigma \in G_N$ must fix all the x_s since $v_E\big(\sigma(x_s) - x_s\big)$ would have to be infinite.)

(c) Fill in the details of the following alternative argument: Since G is finite the G_i become stationary. Furthermore $\bigcap_{i>0} G_i = \{1\}$ since an element thereof acts trivially on $R_E = \varprojlim R_E/\mathfrak{m}_E^{i+1}$. Thus $G_i = \{1\}$ for $i >> 0$.

Problem C. Let $\mathbb{Q}_p^{\mathrm{ur}}$ be the maximal unramified extension of $\mathbb{Q}_p$ in some fixed algebraic closure $\bar{\mathbb{Q}}_p$.

(a) Why is $\mathbb{Q}_p^{\mathrm{ur}}$ <u>not</u> complete? (Use Problem A on HW6.)

(b) Show that its completion $\widehat{\mathbb{Q}_p^{\mathrm{ur}}} \subset \mathbb{C}_p$ has a valuation ring $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}$ which is complete, with p as a uniformizer, and it has residue field[1]

$$\widehat{\mathbb{Z}_p^{\mathrm{ur}}}/(p) \xrightarrow{\sim} \bar{\mathbb{F}}_p.$$

(c) Prove that $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ is topologically generated by Frobenius (i.e., the subgroup generated by the Frobenius automorphism is dense):

$$\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) = \hat{\mathbb{Z}}.$$

(Here $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$ is the profinite completion of the integers.)

Problem D. Let $\mathbb{Q}_p^{\mathrm{tr}} \supset \mathbb{Q}_p^{\mathrm{ur}}$ be the union of all tamely ramified finite extensions of $\mathbb{Q}_p$ in some fixed algebraic closure $\bar{\mathbb{Q}}_p$.

(a) For each $n > 0$ let $\pi_n \in \bar{\mathbb{Q}}_p$ be a root of the polynomial $X^{p^n-1} + p$. Show that $\mathbb{Q}_{p^n}(\pi_n)$ is a totally and tamely ramified degree $p^n - 1$ extension of $\mathbb{Q}_{p^n}$ – which is independent of the choice of root π_n.

(b) Deduce that $\mathbb{Q}_{p^n}(\pi_n)$ is the splitting field of $X^{p^n-1}+p \in \mathbb{Z}_{p^n}[X]$ (therefore Galois) with Galois group

$$\mathrm{Gal}(\mathbb{Q}_{p^n}(\pi_n)/\mathbb{Q}_{p^n}) \xrightarrow{\sim} \mu_{p^n-1}(\bar{\mathbb{Q}}_p) \xrightarrow{\sim} \mathbb{F}_{p^n}^{\times}.$$

(Send σ to the ratio $\frac{\sigma(\pi_n)}{\pi_n}$ and then reduce modulo p.)

[1] Meaning $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}$ is the ring of Witt vectors $W(\bar{\mathbb{F}}_p)$ of the characteristic p perfect field $\bar{\mathbb{F}}_p$.

(c) For $n = 1$ observe that $\mathbb{Q}_p(\pi_1) = \mathbb{Q}_p(\zeta_p)$. (Use Problem B on HW5.)

(d) Verify that $\mathbb{Q}_p^{\mathrm{tr}} = \bigcup_{n>0} \mathbb{Q}_{p^n}(\pi_n)$ and conclude that there is an isomorphism of topological groups

$$\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{tr}}/\mathbb{Q}_p^{\mathrm{ur}}) \xrightarrow{\sim} \varprojlim \mathbb{F}_{p^n}^{\times}$$

where the transition map $\mathbb{F}_{p^n}^{\times} \to \mathbb{F}_{p^m}^{\times}$ is the norm map for $m|n$.

(e) Infer that $P = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{tr}})$ is the unique Sylow pro-p subgroup (meaning the largest pro-p subgroup) of the inertia group $I = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$.

(`Hint`: Let P be any maximal pro-p subgroup. Try to show $\mathbb{Q}_p^{\mathrm{tr}} = \bar{\mathbb{Q}}_p^P$. The inclusion $\subset$ essentially follows from (d). For $\supset$ observe that $\bar{\mathbb{Q}}_p^P$ is the smallest subfield of $\bar{\mathbb{Q}}_p$ with a pro-p Galois group.)