Math 204B, Winter 2019

Number Theory II, HW 9

Due Wednesday March 13th in class (or by noon).

From Neukirch's book `Algebraic Number Theory`:

- Exercises:

  1 on page 181

  (`Hint`: Recall that a $\sigma \in G_0$ lies in G_i if and only if $v\big(\sigma(\Pi)/\Pi - 1\big) \geq i$. Taking $\Pi = \zeta - 1$ reduces the problem to reading off the valuation $v(\zeta^N - 1)$ from the p-expansion of N.)

Problem A. Let K be a field, and let $\mathrm{Gal}_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ be its absolute Galois group (with the Krull topology).

(a) Show that $\mathrm{GL}_n(\mathbb{C})$ has "no small subgroups" – meaning the identity matrix I has an open neighborhood which does not contain any non-trivial subgroup.

(`Hint`: First do this for $\mathbb{C}^\times$. In general, if $\|A - I\| < \epsilon$ then all eigenvalues λ of A satisfy $|\lambda - 1| < \epsilon$. Therefore, if all powers A^N also lie in the ϵ-ball we must have $\lambda = 1$, i.e. A is at least unipotent. However, since the A^N remain bounded we conclude that $A = I$.)

(b) Deduce from (a) that any continuous representation $\mathrm{Gal}_K \to \mathrm{GL}_n(\mathbb{C})$ factors through $\mathrm{Gal}(E/K)$ for some finite Galois extension E/K. When $n = 1$ check that one can take E/K to be an abelian extension.

(c) Let $n = 1$ and suppose K is a non-archimedean local field. Explain why composition with the Artin map ϕ_K defines a bijection

$$\{\text{continuous characters } \mathrm{Gal}_K \to \mathbb{C}^\times\} \overset{1:1}{\longleftrightarrow}$$
$$\{\text{continuous characters } K^\times \to \mathbb{C}^\times \text{ of } \textbf{finite} \text{ order}\}.$$

(It sends $\chi \mapsto \chi^{\mathrm{ab}} \circ \phi_K$ where χ^{ab} is the character of $\mathrm{Gal}_K^{\mathrm{ab}}$ given by χ.)

Problem B. Let $K/\mathbb{Q}_p$ be a finite extension, with absolute Galois group Gal_K. The cyclotomic character $\chi_{\mathrm{cyc}} : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is the projection onto

$$\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) = \varprojlim \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) \simeq \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times.$$

Its restriction to $\mathrm{Gal}_K \subset \mathrm{Gal}_{\mathbb{Q}_p}$ will also be denoted by χ_{cyc}.

(a) Check that $\chi_{\mathrm{cyc}} : \mathrm{Gal}_K \to \mathbb{Z}_p^\times$ is continuous, but <u>not</u> of finite order.

(b) Consider the composition $\chi_{\mathrm{cyc}}^{\mathrm{ab}} \circ \phi_K$, which is the character $K^\times \to \mathbb{Z}_p^\times$ corresponding to χ_{cyc} via class field theory. Verify that

$$(\chi_{\mathrm{cyc}}^{\mathrm{ab}} \circ \phi_K)(x) = N_{K/\mathbb{Q}_p}(x) \cdot \|x\|_K \qquad \forall x \in K^\times$$

where $\| \cdot \|_K$ is the normalized absolute value on K.

(`Hint`: Reduce to the case $K = \mathbb{Q}_p$ utilizing that Artin maps are compatible with norm maps. To see that $p \mapsto 1$ write p as a norm from $\mathbb{Q}_p(\zeta_{p^n})$. Finally check that a unit $u \in \mathbb{Z}_p^\times$ is mapped to itself using $\mathbb{Z}_p^\times \xrightarrow{\sim} I_{\mathbb{Q}_p}^{\mathrm{ab}}$.)

Problem C. Let K be a non-archimedean local field. The Weil group $W_K \subset \mathrm{Gal}_K$ consists of the automorphisms which act as $\mathbb{Z}$-powers of Frobenius on the residue field. Thus it sits in a short exact sequence

$$0 \longrightarrow I_K \longrightarrow W_K \longrightarrow \mathbb{Z} \longrightarrow 0 \qquad (1)$$

where $I_K = \mathrm{Gal}_{K^{\mathrm{ur}}}$ is the inertia subgroup. (Compare this to the short exact sequence

$$0 \longrightarrow I_K \longrightarrow \mathrm{Gal}_K \longrightarrow \hat{\mathbb{Z}} \longrightarrow 0$$

where $\hat{\mathbb{Z}} \simeq \mathrm{Gal}_k = \overline{\langle \mathrm{Frob} \rangle}$ is the absolute Galois group of the residue field k.)

(a) Endow I_K with the Krull topology. Prove that there is a unique topology on W_K which makes (1) a short exact sequence of **topological** groups (meaning all maps are continuous and $W_K/I_K \xrightarrow{\sim} \mathbb{Z}$ is a homeomorphism.)

(`Hint`: As a neighborhood basis at the identity take all $\mathrm{Gal}_E \subset I_K$ where E/K^{ur} is a varying finite extension.)

(b) Show that W_K is a dense subgroup of Gal_K, but the topology on W_K defined in (a) is **stronger** than the induced topology from Gal_K.

(c) Verify that the Artin map ϕ_K defines a topological isomorphism

$$K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}.$$

(Here $K^\times$ carries the standard topology defined by $\| \cdot \|_K$.)

Problem D. Thank you all for a great quarter! Please fill out your CAPE teaching evaluations (due Monday March 18th at 8AM).