WEEK 2.

Therefore,

$$|a| = |p|^{v_p(a)} \cdot 1 = |a|_p^c$$

where $|p| = p^{-c}$, i.e. $c = \log_p\left(\frac{1}{|p|}\right) > 0$.

By multiplicativity $|a| = |a|_p^c$ for all $a \in \mathbb{Q}^\times$.

I.e., $|\cdot| = |\cdot|_p^c$. ✓

(2) Suppose $|\cdot|$ is <u>archimedean</u>, and <u>not</u> $\sim |\cdot|_\infty$.

Know (criterion for $\sim$):

$$\exists\, b \in \mathbb{Q} \quad \text{s.t.} \quad |b| < 1 \text{ and } |b|_\infty \geq 1.$$

(take $b > 0$)

$$\hookrightarrow \text{so } |b| < 1 \text{ and } b > 1.$$

<u>EXC</u> ("expansion in base b") $b = \frac{c}{d} > 1$. Every $n \in \mathbb{N}$ can be written

$$n = a_0 + a_1 b + \dots + a_r b^r \quad \text{where } 0 \leq a_i < c.$$

(Hint: induction on n. $n = mc + a_0$, $0 \leq a_0 < c$.

$\Rightarrow dn = (dm)c + a_0 d$ with $dm \leq d\frac{n}{c} < n$

Thus, by ind., $dm = a_1 + a_2 b + \dots + a_r b^{r-1}$ ← since $b > 1$.

insert and div. by d.)

<u>obs</u>: $|a| = |\underbrace{1 + \dots + 1}_{a}| \leq \underbrace{|1| + \dots + |1|}_{a} = a \quad \forall a \in \mathbb{N}.$

– Expand an $n \in \mathbb{N}$ as in above exc. Then,

$$|n| \leq |a_0| + |a_1| \cdot |b| + \cdots + |a_r| \cdot |b|^r$$

$$\leq c + c|b| + \cdots + c|b|^r = c \, \frac{1 - |b|^{r+1}}{1 - |b|} < \frac{c}{1 - |b|}$$

Shows $|\mathbb{Z}|$ bounded; contradiction. $\square$

<u>Ostrowski</u> ($K = k(t)$): Recall our examples:

k ← <u>any</u> field.

(i) $\forall \mathfrak{p} \subseteq k[t]$ max, $|f|_{\mathfrak{p}} = q^{-v_{\mathfrak{p}}(f)}$

(ii) $|f|_\infty = q^{\deg(f)}$

(any choice $q > 1$). so up to $\sim$ only...

∘ <u>Theorem</u>:

– Up to equivalence these are the only non-trivial $|\cdot|$ on $k(t)$, which are <u>trivial</u> on k. (automatic if $|k| < \infty$)

<u>Why</u>? Given $|\cdot|$ non-triv., but <u>trivial</u> on $k \subseteq K$. ($\Rightarrow$ non-arch., since $\mathbb{Z} \cdot 1_K \subseteq k$)

– Consider $|t|$

either ≤ 1 (show $\sim |\cdot|_{\mathfrak{p}}$) or > 1 ($\sim |\cdot|_\infty$)

(1) Suppose $|t| \leq 1$.

Thus $|f| \leq 1$, $\forall f \in k[t]$.

introduce:

$I = \{f : |f| < 1\}$

ideal in $k[t]$, $I \neq 0$ since $|\cdot|$ non-trivial.

I prime ideal: $|fg|<1 \Rightarrow |f|<1$ or $|g|<1$.

Rename it $\mathfrak{p} = (p)$. ← monic irreducible.

Factor

$$f = p^{v_{\mathfrak{p}}(f)} u \quad \text{where} \quad p \nmid u.$$

← equivalent: $u \notin \mathfrak{p}$ i.e., $|u| = 1$.

Then

$$|f| = |p|^{v_{\mathfrak{p}}(f)} = |f|_{\mathfrak{p}}^{c}$$

where $|p| = q^{-c}$, i.e. $c = \log_q\left(\frac{1}{|p|}\right) > 0$. ✓

(2) Now suppose $|t| > 1$.

Then $1 < |t| < |t|^2 < \cdots$

$k[t] \ni f = a_0 + a_1 t + \cdots + a_r t^r$ ← nonzero.

$|f|_\infty = q^r$

Here $|a_i| = \begin{cases} 1 \\ 0 \end{cases}$ since $|\cdot|$ triv. on k.

Note: $|a_i t^i| < |a_r t^r|$ for all $i \neq r$.

$$|a_r t^r| = |t|^r$$

so "max is achieved once":

$$|f| = \max\{|a_i t^i| : i = 0, \ldots, r\} = |t|^r = |f|_\infty^c$$

if we take $c > 0$ s.t.

$|t| = q^c$, i.e. $c = \log_q |t| > 0$. $\square$

Completions. $(K, |\cdot|)$ is complete if Cauchy sequences converge.

— in general:

convergent $\Rightarrow$ Cauchy $\Rightarrow$ bounded

Ex $\mathbb{R}$ or $\mathbb{C}$ with $|\cdot|_\infty$ are complete ($\mathbb{Q}$ isn't)

Useful obs.: If K complete & non-archimedean,

$$\sum_{n=0}^{\infty} a_n \text{ converges } \underline{\text{iff}} \quad a_n \longrightarrow 0.$$

($\Rightarrow$ obvious) $\Leftarrow$: Partial sums $s_n = \sum_{i=0}^{n} a_i$.

(s_n) Cauchy:

$$s_m - s_n = a_{n+1} + \cdots + a_m$$

(say $m > n$) so

$$|s_m - s_n| \leq \max\{\underbrace{|a_{n+1}|, \ldots, |a_m|}_{\text{all } < \varepsilon \text{ for } n \geq N.}\}$$

$\square$

— For instance, if we let

$$R = \{x \in K : |x| \leq 1\}$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < 1$.

Evaluate formal power series on open unit ball at 0:

$$\phi_x : R[[X]] \longrightarrow R \qquad \text{for } |x| < 1.$$
$$X \longmapsto x \qquad \text{(surj. } R\text{-alg. hom.)}$$

K with $|\cdot|$.

<u>Def.</u> A "<u>completion</u>" is pair $(\widehat{K}, \|\cdot\|)$ where

- $\widehat{K}$ <u>field</u> w. embedding $\iota : K \longrightarrow \widehat{K}$.
- $\|\cdot\|$ abs. value on $\widehat{K}$ extending $|\cdot|$.

<u>satisfying</u>: (more precisely $\|\iota(x)\| = |x|$ for $x \in K$)

(i) $\widehat{K}$ with $\|\cdot\|$ is <u>complete</u>,

(ii) K is <u>dense</u> in $\widehat{K}$ (rather $\iota(K) \subseteq \widehat{K}$)

<u>Theorem</u>. Given $(K, |\cdot|)$.

(1) There <u>exists</u> a completion.

(2) If $(K_1, \|\cdot\|_1)$ and $(K_2, \|\cdot\|_2)$ are two completions,

$$\exists ! \ K\text{-isomorphism (fields)} \quad \varphi : K_1 \xrightarrow{\sim} K_2$$

which is <u>isometric</u>, i.e. $\|\underset{\substack{\cap \\ K_1}}{\varphi(x)}\|_2 = \|x\|_1$.

<u>Ex</u> $\mathbb{Q}_p$ is "the" completion of $(\mathbb{Q}, |\cdot|_p)$.

- Uniqueness? Suppose $(\hat{K}, \|\cdot\|)$ is a completion, E any complete field w. $|\cdot|_E$.

> CLAIM: Any isometric embedding $K \xrightarrow{i} E$ extends uniquely to $\hat{K} \xrightarrow{I} E$, isometric.

(I unique: $K \subseteq \hat{K}$ dense)

— How? Given $x \in \hat{K}$ find (x_n) in K converging to x.

Note: $(i(x_n))$ Cauchy in E ($\Rightarrow$ convergent)

$$|i(x_m) - i(x_n)|_E = |x_m - x_n| < \varepsilon \quad \forall m,n \geq N.$$

Let $I(x) := \lim_{n\to\infty} i(x_n)$ [check indep. of choice $x_n \to x$].

$I: \hat{K} \to E$ embedding (since i is), and

$$\|I(x)\|_E = \lim_{n\to\infty} |i(x_n)|_E = \lim_{n\to\infty} |x_n| = \|x\|$$

(isometric) ✓

I extends i: Take constant sequence $x_n = x$.

FINALLY,

– If E is another completion, get K-linear isometric cont. embeddings $\hat{K} \to E$ and $E \to \hat{K}$ (mutually inverse since $K \subseteq \hat{K}$ dense).

∘ Existence? (sketch)

$C = \{\text{Cauchy seq. in } K\}$ comm. ring.

$N = \{\text{Null seq. in } K\}$ ideal, max ← exc.

$\hat{K} = C/N$ field, embedding $\iota: K \longrightarrow \hat{K}$ "constant seq."

↳ abs. value $\|\cdot\|$?

$x \in \hat{K}$ rep. by (x_n).

$\|x\| := \lim_{n\to\infty} |x_n| \in [0,\infty)$. [uses $\mathbb{R}$ complete].

Exc Show this works: $K \subseteq \hat{K}$ dense, $\hat{K}$ complete.
– omit, cf. [Lorenz].

Prop. 0) K non-archimedean, so is $\hat{K}$. – bounded on ($\mathbb{Z}\cdot 1_K = \mathbb{Z}\cdot 1_{\hat{K}}$)

1) Same valuation group $\leq \mathbb{R}_+^\times \cong \mathbb{R}$:

(assuming non-arch.) $|K^\times| = |\hat{K}^\times|$

2) Let R and $\hat{R}$ be the unit (closed) balls, $R = \{x \in K : |x| \leq 1\}$ etc.
(assume non-arch.)

$\mathfrak{m} = \{x \in K : |x| < 1\}$
(same: $\hat{R}, \hat{\mathfrak{m}}$)

– Same residue fields.

$$R/\mathfrak{m} \xrightarrow{\sim} \hat{R}/\hat{\mathfrak{m}}.$$

Ex: $|\mathbb{Q}^\times|_\infty = \mathbb{Q} \cap (0,\infty)$ $\quad |\mathbb{R}^\times|_\infty = (0,\infty)$

PF. (1): $x \in \widehat{K}^{\times}$. Choose $\overset{\substack{K \\ \rotatebox{90}{$\in$}}}{x_n} \to x$. Then $|x - x_n| < |x|$, for $n >> 0$.

so $\underset{\substack{\rotatebox{-90}{$\in$} \\ |K^{\times}|}}{|x_n|} = \max\{|x - x_n|, |x|\} = |x|$.

(2). Obviously $R \cap \widehat{m} = m$. Surjective: $x \in \widehat{R}$. By density $x + \widehat{m}$ contains some $y \in K$.

$$|y| \leq \max\{|x|, \underbrace{|y - x|}_{<1}\} \leq 1 \text{ so } y \in R. \qquad \square$$

<u>Setup</u>: K any field w. non-arch. $|\cdot|$ (not nec. complete)

$R = \mathcal{O}_K = \{x \in K : |x| \leq 1\}$ subring.

$\mathfrak{m} = \mathfrak{m}_K = \{x \in K : |x| < 1\}$ ideal

<u>note</u>: $R^\times = \{x \in K : |x| = 1\} = R \setminus \mathfrak{m}$,

– i.e. R is a <u>local</u> domain w. <u>max</u> ideal $\mathfrak{m}$.

- "<u>residue field</u>" $k = k_K = R/\mathfrak{m}$.
- "<u>fraction field</u>" $\mathrm{Frac}(R) = K$

<u>Ex</u> ($K = \mathbb{Q}$ with $|\cdot|_p$) Here $R = \mathbb{Z}_{(p)}$, $\mathfrak{m} = p\mathbb{Z}_{(p)}$, $k = \mathbb{F}_p$.

<u>Obs</u>: R <u>integrally closed</u> in K ("normal domain")

– Why? Suppose $x \in K$ sat. $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ (where all $a_i \in R$)

Then,

$$|x|^n \leq \max\{|a_0|, |a_1 x|, \ldots, |a_{n-1}x^{n-1}|\}$$

$$\leq \max\{1, |x|, \ldots, |x^{n-1}|\} = |x|^{n-1}$$

if $|x| > 1$

Contradiction. So $x \in R$ ✓

<u>Def</u>. A non-trivial $|\cdot|$ on K is "<u>discrete</u>" if the valuation group $|K^\times| \subseteq \mathbb{R}_+^\times$ is discrete.

– i.e., $|K^\times| = c^{\mathbb{Z}}$ for some $c > 0$. We'll take $c < 1$.

($\Longrightarrow$ <u>non-archimedean</u>: Otherwise $\mathbb{Q} \subseteq K$ and $|\cdot| \sim |\cdot|_\infty$ on $\mathbb{Q}$)

— A "uniformizer" is $\pi \in K$ with $|\pi| = c$ (thus $\pi \in \mathfrak{m}$ and $|K^\times| = |\pi|^{\mathbb{Z}}$).

unique up to units.

EXC $\mathfrak{m} = (\pi) = \pi R$

∘ Valuation: $x \in K^\times$ for $|x| = c^{v(x)}$

($v : K^\times \to \mathbb{Z}$) — i.e., $x = u\pi^{v(x)}$

$|u| = 1$.

— Choice of π gives isom. of groups:

$K^\times \simeq \mathbb{Z} \times R^\times$ (non-can.)

$x \longmapsto (v(x), u)$.

— EXC (cont)

1) (π^r) for $r \geq 0$ are all the nonzero ideals in R. Thus R is PID ($\Longrightarrow$ Noetherian)

(Hint: $I \neq 0$. Pick nonzero $x \in I$ with $v(x) \geq 0$ minimal. Then $I = (x) = (\pi^{v(x)})$.)

2) $(\pi^r) = \mathfrak{m}^r = \{x \in K : |x| \leq c^r\}$ closed + open.

$= \{x \in K : |x| < c^{r-1}\}$

neighborhood basis at 0.

In fact R is a DEDEKIND domain. — follows.

(Noetherian, normal, dim = 1)

K with discrete $|\cdot|$. Completion: $\hat{K}$ with $\|\cdot\|$ (corresponding $\hat{m} \subseteq \hat{R}$ etc.)

same res. fields. ~ Strengthening:

Since $|\hat{K}^\times| = |K^\times|$, $\|\cdot\|$ still discrete and π remains a uniformizer in $\hat{K}$.

Thm.

$$R/m^r \xrightarrow{\sim} \hat{R}/\hat{m}^r$$

for all $r>0$.

Pf. $R \cap \hat{m}^r = \{x \in R : \|x\| \leq c^r\} = m^r$.

Surjective: $R \subseteq \hat{R}$ dense so given $x \in \hat{R}$, $R \cap (x + \hat{m}^r) \neq \emptyset$. $\square$

— note: Continuous injective hom. rings

$$\psi : R \longrightarrow \varprojlim R/m^r$$

w. dense image (check)

~ in particular: ψ is surjective if K complete.

ψ is an open map in general (so homeomorphism onto its image). $\quad \psi(m^s) = (\underbrace{00\ldots0}_{s} ** \ldots)$ (image of mult. by π^s)

Upshot:

$$\varprojlim R/m^r \xrightarrow{\sim} \varprojlim \hat{R}/\hat{m}^r \xleftarrow{\sim} \hat{R}$$

(so $\hat{R}$ is the alg. m-adic completion of R)

Compactness criterion: $\hat{R}$ compact $\Longleftrightarrow$ k finite.

Pf. $\Longrightarrow$: Pick representatives $S \subseteq R$ for k.

$$\hat{R} = \bigodot_{x \in S} x + \hat{m} \quad \text{has } \underline{\text{finite}} \text{ subcover.}$$

disjoint, so must have $|S| < \infty$.

$\Longleftarrow$: First obs. that $|R/m^r| = q^r$ where $q = |k|$.

Indeed, filtration:

$$R \supseteq (\pi) \supseteq (\pi^2) \supseteq \cdots \supseteq (\pi^r)$$

and $R/m \xrightarrow{\sim} (\pi^r)/(\pi^{r+1})$ of groups.

$$x + m \longmapsto \pi^r x + (\pi^{r+1})$$

Thus $\hat{R} \simeq \varprojlim \underbrace{R/m^r}_{\text{finite}}$ is even $\underline{\text{profinite}}$ ($\overset{\text{Tychonoff}}{\Longrightarrow}$ compact)

$\square$

* Deduce: When k finite, $\hat{K}$ is locally compact.

($\rightsquigarrow$ harmonic analysis)