WEEK
3.

<u>Converse</u>: K with $|\cdot|$ non-archimedean. (non-trivial)

If K <u>locally</u> <u>compact</u>, then: (1) k finite
(2) $|K^\times|$ discrete
(3) K complete.

"<u>local</u> <u>field</u>" (non-arch.)

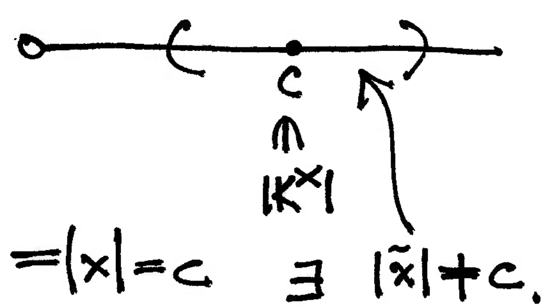
[i.e. every $x \in K$ admits open neighborhood $U \ni x$ contained in a compact set: $x \in U \subseteq C$.]

equivalently (exc): closed balls are compact, ex. R.

<u>KNOW</u>: R compact $\Longleftrightarrow$ k finite.

- Completeness (3): (x_n) in K Cauchy, then bounded $\subseteq \overline{B_r(0)}$ for $r \gg 0$. By compactness $\exists$ conv. subseq. x_{n_i}. Being Cauchy, x_n itself conv.

- Discreteness (2): Otherwise there's some $c > 0$ admitting a sequence $|x_n| \to c$ w. <u>distinct</u> terms. (x_n) bounded. As above, extract $x_{n_i} \to x$. (has $|x| = c$). However,



$$|x_n| = \max\{|x|, \underbrace{|x_n - x|}_{\substack{\text{small} \\ < |x|}}\} = |x| = c$$

for $n \gg 0$. contradiction. ✓

~ LATER:

Classification: If K admits a non-trivial $|\cdot|$ s.t. K is locally compact ("local field") then

1) $K = \mathbb{R}$ or $K = \mathbb{C}$ — archimedean case,
2) $K/\mathbb{Q}_p$ finite ext.
3) $K = k((t))$ for finite k.

(and conv.) — equivalently $K/\mathbb{F}_p((t))$ fin. ext.

GLOBAL fields are:
- number fields; fin. $K/\mathbb{Q}$
- function fields; fin. $K/\mathbb{F}_p(t)$.

— will show all local fields are completions of global fields, and vice versa.

~ Back to non-arch. locally compact $(K, |\cdot|)$:

Def. The "normalized" abs. value on K is

$$|x|_K = q^{-v_K(x)} \quad \text{where} \quad q = |k|$$

(and $v_K : K^\times \to \mathbb{Z}$ valuation $v_K(u\pi^v) = v$)

Note: Balls $R, \mathfrak{m}$ only dep. $|\cdot|$ up to equivalence. "uniformizer" π is a generator for $\mathfrak{m}$.

K with <u>discrete</u> $|\cdot|$, residue field $k = R/m$

Pick <u>uniformizer</u> π. — choose <u>representatives</u> $S \subseteq R$

$m = (\pi)$ (i.e. $S \to R/m$ bijection)

<u>EX</u> $K = \mathbb{Q}$ with $|\cdot|_p$.

$S = \{0, 1, 2, \ldots, p-1\}$

<u>EX</u> $K = k(t)$ with $|\cdot|_\infty$

Here $|\cdot|_t$ $\quad |f|_t = q^{-v_t(f)}$

$R = k[t]_{(t)}$ has <u>residue field</u> k

— may take $S = k \subseteq K$

"constants"

[closed under $+$ and $\cdot$]

Given $x \in R$. Pick <u>the</u> $a_0 \in S$ with $x \equiv a_0 \pmod{m}$.

write $x = a_0 + \pi x_1$.

— Continue:

$$\begin{cases} x_1 = a_1 + \pi x_2 \\ x_2 = a_2 + \pi x_3 \\ \vdots \end{cases}$$

Combined: $\forall n$,

$$x = a_0 + \pi(a_1 + \pi x_2) = a_0 + a_1\pi + \pi^2 x_2 = \ldots$$

$$a_0 + a_1\pi + \cdots + a_n\pi^n + \underbrace{\pi^{n+1} x_{n+1}}_{\downarrow \, 0}$$

- Shows $x \in R$ has a unique expansion: $x = \sum_{n=0}^{\infty} a_n \pi^n$ with all $a_n \in S$.

Conversely, if K complete, every such sum converges. Thus any $x \in K$ is given by a "Laurent series":

$$x = \sum_{n=-N}^{\infty} a_n \pi^n \quad \text{with } a_n \in S$$

(and vice versa if K complete).

EX. Elements of $\mathbb{Z}_p$ are $x = \sum_{n=0}^{\infty} a_n p^n$ with $0 \le a_n < p$.
evaluation at p, (uncountable!)

$$\varphi : \mathbb{Z}[[X]] \twoheadrightarrow \mathbb{Z}_p \quad \text{has} \quad \ker(\varphi) = (X - p)$$
$$X \longmapsto p \qquad\qquad (\subseteq : \dim = 2)$$

EX. Elements of $\widehat{k(t)}$ (completion rel. $|\cdot|_t$) are formal Laurent series $\sum_{n \gg -\infty}^{\infty} a_n t^n$ with $a_n \in k$.
Even have: $\widehat{k(t)} \simeq k((t))$ since in this ex. S closed under $+$ and $\cdot$.

[in general, adding/mult. π-expansions complicated].

§ Haar measure: G locally compact Hausdorff top. group.
(ex. K with $+$)

"Borel sets:" σ-alg. generated by open subsets.

$\parallel$

Σ

closed under complement, countable unions & intersections.
(contains $\emptyset$ and G)

A measure (Borel) is $\mu : \Sigma \longrightarrow \mathbb{R} \cup \{\infty\}$ s.t.

a) $\mu(E) \geq 0 \quad \forall E \in \Sigma$.

b) $\mu(\emptyset) = 0$

c) countably additive, i.e.

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

for $E_1, E_2, \ldots$ pairwise disjoint.

Thm (HAAR) There's a Borel non-trivial measure μ on G:

i) Translation-invariant: $\mu(xE) = \mu(E)$ all $x \in G$.

ii) $\mu(E) < \infty$ for compact E.

iii) Outer regular: $\forall E \in \Sigma$,

$$\mu(E) = \inf\{\mu(U) : U \supseteq E \text{ open}\}$$

iv) Inner regular: $\forall$ open E,

$$\mu(E) = \sup\{\mu(C) : C \subseteq E \text{ cpt.}\}$$

Such μ is unique up to $c > 0$.

$\underline{\text{Ex}}$ $(K = \mathbb{R}_{\text{add.}})$ $\mu =$ Lebesgue measure, $\mu(x+E) = \mu(E)$.

— get μ on our (non-arch.) K $\underline{\text{local field}}$ (add. group)

Let $x \in K^\times$. Consider $E \overset{\mu_x}{\longmapsto} \mu(x \cdot E)$

Σ — another Haar measure.

By $\underline{\text{uniqueness}}$, there's $c_x > 0$: (translation-invt.? $\mu_x(a+E) = \mu(xa + xE) = \mu(xE) = \mu_x(E)$ ✓)

$$\mu_x(E) = \mu(x \cdot E) = c_x \, \mu(E), \quad \forall E \in \Sigma.$$

$\underline{\text{CLAIM}}$: $c_x = |x|_K$ (= normalized abs. value)

i.e., $\mu(xE) = |x|_K \, \mu(E)$.

Why? 1st check

$\|x\| := c_x$ def. an abs. value on K.

· multiplicative ✓ · strong triangle ineq.:

$$\|x+y\| \, \mu(E) = \mu((x+y)E)$$

$$\leqslant \mu(xE) = \|x\| \, \mu(E).$$

— take $E = R$ here

By $\underline{\text{regularity}}$ $\mu(E) \neq 0$. Thus

$$\|x+y\| \leqslant \|x\| = \max\{\|x\|, \|y\|\}$$

✓

Suppose WLOG $|x| \geqslant |y|$, i.e. $y \in xR$. Thus $yR \subseteq xR$ implies $\mu(yR) \leqslant \mu(xR)$ i.e., $\|y\| \leqslant \|x\|$.

REMAINS: $\|\pi\| \overset{?}{=} q^{-1}$ (i.e., $\mu(\pi R) \overset{?}{=} q^{-1}\mu(R)$)

Note $\pi R = m$, so $R = \bigsqcup_{x \in S} x + m$ ← representatives for k, $|S| = q$.

$\Rightarrow \mu(R) = \sum_{x \in S} \mu(x + m)$

$= \sum_{x \in S} \mu(m) = q\,\mu(\pi R)$ ✓.

Remark:

- For $\mathbb{C}$: $\mu(xE) = |x|_\infty^2 \cdot \mu(E)$.

($E \subseteq \mathbb{C}$ cpt. ball say at 0.) ← NOT an abs. value.

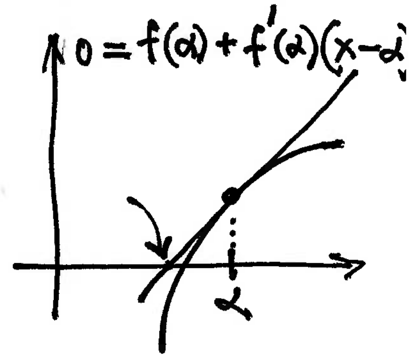
xE scale radius by $|x|_\infty$ = modulus.

<u>Newton's Method</u>: K with non-arch. $|\cdot|$, <u>complete</u>, $R \supset m$.

$f(x) = a_0 + a_1 x + \cdots + a_N x^N \in R[x]$ (not nec. discrete)

Suppose $\alpha_0 \in R$ sat. $|f(\alpha_0)| < |f'(\alpha_0)|^2$.

<u>Def.</u> $\alpha_{n+1} = \alpha_n - \dfrac{f(\alpha_n)}{f'(\alpha_n)}$ $(\forall n \geq 0)$ cf. "Newton"



<u>CLAIM</u>: Well-defined, i.e. all $f'(\alpha_n) \neq 0$.

— In fact, $|f'(\alpha_n)| = \cdots = |f'(\alpha_0)|$.

More importantly:

- (α_n) <u>converges</u>, and $\alpha := \lim_{n\to\infty} \alpha_n \in R$ is a <u>root</u> of f. $(f(\alpha) = 0)$

- $|\alpha - \alpha_0| \leq \left|\dfrac{f(\alpha_0)}{f'(\alpha_0)^2}\right| < 1$.

<u>Why</u>? Let $c = \left|\dfrac{f(\alpha_0)}{f'(\alpha_0)^2}\right| < 1$.

— By induction we'll show:

(i) $|\alpha_n| \leq 1$

(ii) $|\alpha_n - \alpha_0| \leq c$

(iii) $|f(\alpha_n)| \leq c^{2^n} |f'(\alpha_n)|^2$

$(n = 0$ ok$)$.

$|f'(\alpha_n)| \leq 1.$

Assume $n \geq 0$ and ok for n.

First, knowing $\left|\frac{f(\alpha_n)}{f'(\alpha_n)^2}\right| \leq c^{2^n}$ gives (by def. α_{n+1}):

$$\boxed{|\alpha_{n+1} - \alpha_n| \leq c^{2^n}}$$

— in particular this is <1 and thus $|\alpha_{n+1}| \leq 1$: (i) ✓ (since $|\alpha_n| \leq 1$)

— For (ii) note:

$$|\alpha_{n+1} - \alpha_0| \leq \max\{\underbrace{|\alpha_{n+1} - \alpha_n|}_{\leq c^{2^n}}, \underbrace{|\alpha_n - \alpha_0|}_{\leq c}\} = c.$$

(ii) ✓

— Finally (iii): Taylor,

$$f(\alpha_{n+1}) = \underbrace{f(\alpha_n) - f'(\alpha_n)\cdot\frac{f(\alpha_n)}{f'(\alpha_n)}}_{0} + \underbrace{c_2\left(\frac{f(\alpha_n)^2}{f'(\alpha_n)^2}\right) + \cdots}_{\beta\cdot\left(\frac{f(\alpha_n)}{f'(\alpha_n)}\right)^2}$$

for some $\beta \in R$.

Thus, as $\beta \in R$,

$$*)\ |f(\alpha_{n+1})| \leq \left|\frac{f(\alpha_n)}{f'(\alpha_n)}\right|^2 =: \varepsilon^2$$

[note: $\frac{f(\alpha_n)}{f'(\alpha_n)} = \underset{\substack{\cap \\ R}}{\frac{f(\alpha_n)}{f'(\alpha_n)^2}} \cdot \underset{\substack{\cap \\ R}}{f'(\alpha_n)}$]

shows $\varepsilon \leq c^{2^n}$.

Combine this with Taylor for f':

$$f'(\alpha_{n+1}) = \underbrace{f'(\alpha_n)}_{\text{abs} > \varepsilon} - \underbrace{f''(\alpha_n)\cdot\frac{f(\alpha_n)}{f'(\alpha_n)}}_{\text{abs} \leq \varepsilon} + \underbrace{\cdots\cdots}_{\text{abs} \leq \varepsilon^2}$$

see next page.

Fist term has $|f'(\alpha_n)| \overset{(iii)}{\geqslant} c^{-2^n} \cdot \underbrace{\left|\frac{f(\alpha_n)}{f'(\alpha_n)}\right|}_{\varepsilon} = \varepsilon \cdot c^{-2^n} > \varepsilon.$ $(c<1)$

Deduce

$$|f'(\alpha_{n+1})| = |f'(\alpha_n)|.$$

Conclude (w. (*)) that

$$\left|\frac{f(\alpha_{n+1})}{f'(\alpha_{n+1})^2}\right| \leqslant \left|\frac{f(\alpha_n)}{f'(\alpha_n)}\right|^2 \cdot \frac{1}{|f'(\alpha_n)|^2} = \left|\frac{f(\alpha_n)}{f'(\alpha_n)^2}\right|^2 \leqslant c^{2^{n+1}}$$

Shows (i) —— (iii).

(iii) ✓

→ Back to claim: $\boxed{}$ shows (α_n) is Cauchy.

By (iii), $\alpha \overset{df.}{=} \lim \alpha_n$ (K complete)

$|f(\alpha_n)| \leqslant c^{2^n}$ $\quad \in R$

so $f(\alpha_n) \longrightarrow f(\alpha) = 0$. root.

At last, checked $|\alpha_n - \alpha_0| \leqslant c$ in (ii). Let $n \to \infty$:

$|\alpha - \alpha_0| \leqslant c.$

SUMMARY: K complete (non-arch.).
$f \in R[X]$. Suppose $\alpha_0 \in R$ s.t. $|f(\alpha_0)| < |f'(\alpha_0)|^2$.
Then $\exists \alpha \in R$ s.t.
(1) $f(\alpha) = 0$
(2) $|\alpha - \alpha_0| \leqslant \left|\frac{f(\alpha_0)}{f'(\alpha_0)^2}\right| < 1.$