WEEK
3.
Converse: $K$ with $1:1$ non-archimedean (non-trivial)

If $K$ locally compact, then:

1. $k$ finite
2. $1 \times 1$ discrete
3. $K$ complete

[i.e. every $x \in K$
admits open neighborhood $U \ni x$
which is contained in a compact set:
$x \in U \subseteq C$.]

equivalently: closed balls are compact, e.g., $\mathbb{R}$.

Know: $\mathbb{R}$ compact $\iff k$ finite.

Completeness:

1. $\{x_n\}$ in $K$ Cauchy, then bounded $\subseteq B_r(0)$
2. By compactness $\exists$ conv. subseq. $x_{n_i}$. Being Cauchy, $x_n$ itself conv.

Discreteness:

Otherwise, there's some $c > 0$ admitting
a sequence $\{x_n\} \to c$ w. distinct terms,
$(x_n)$ bounded. As above,
extract $x_{n_i} \to x$.

$(\exists x|c|)$. However,

$\|x_n\| = \max\{\|x_1\|, \|x_n - x_1\| \leq \|x_1 - c\| \implies \|x_n\| < \|x_1\| + c\}$

for $n \gg 0$. small contradiction. $\checkmark$
LATER:

Classification: If $K$ admits a non-trivial $|.|$
so $K$ is locally compact ("local field") then
1) $K = \mathbb{R}$ or $K = \mathbb{C}$ — archimedean case,
2) $K/\mathbb{Q}$ finite ext.
3) $K = k((t))$ for finite $k$.

(and conv.) equivalently $K/\mathbb{F}_p((t))$ fin. ext.

GLOBAL fields are:
- number fields: fin. $K/\mathbb{Q}$
- function fields: fin. $K/\mathbb{F}_p(t)$

will show all local fields are completions of global fields, and vice versa.

Back to non-arch. locally compact $(K, |.|)$:

Def. The "normalized" abs. value on $K$ is

$$|x|_K = q^{-v_K(x)}$$

where $q = |k|$ (and $v_K: K^\times \rightarrow \mathbb{Z}$ valuation $v_K(x\pi^v) = v$)

Note: Balls $B_1, m$ only dep. $1/\pi$ up to equivalence.
"uniformizer" $\pi$ is a generator for $m$. 
Given $x \in R$. Pick the $a_0 \in S$ with $x \equiv a_0 \pmod{m}$.

Continue:

$$\begin{cases} x_1 &= a_1 + \pi x_2 \\ x_2 &= a_2 + \pi x_3 \\ \vdots & \end{cases}$$

Combined, $\forall n$,

$$x = a_0 + \pi (a_1 + \pi x_2) = a_0 + a_1 \pi + \pi^2 x_2 = \cdots$$

$$a_0 + a_1 \pi + \cdots + a_n \pi^n + o \pi^{n+1} x_{n+1} \downarrow 0$$
Show $x \in K$ has a unique expansion:

$$x = \sum_{n=0}^{\infty} a_n \pi^{-n} \text{ with all } a_n \in S.$$  

Conversely, if $K$ complete, every such sum converges.
Thus any $x \in K$ is given by a "Laurent series":

$$x = \sum_{n=-\infty}^{\infty} a_n \pi^{-n} \text{ with } a_n \in S$$

(and vice versa if $K$ complete).

Ex. Elements of $\mathbb{Z}_p$ are $x = \sum_{n=0}^{\infty} a_n p^n$ with $0 \leq a_n < p$. Evaluation at $p$:

$q: \mathbb{Z}[[X]] \to \mathbb{Z}_p$ has $\ker(q) = (X-p) \quad \text{ (uncountable!)}$

$$X \mapsto p \quad \text{ (} \leq \text{ dim}=2\text{)}$$

Ex. Elements of $\widehat{k(t)}$ (completion rel. $1-t$) are formal Laurent series $\sum_{n=-\infty}^{\infty} a_n t^n$ with $a_n \in k$. Even here: $\widehat{k(t)} \cong k((t^\infty)$ since in this ex. $S$ closed under $+$ and $\cdot$.

[In general, adding/multiplying $\pi$-expansions complicated.]
Haar measure \( \mu \) on a locally compact Hausdorff top. group \( G \) (e.g., \( K \) with +)

- **Borel sets:** \( \Sigma \) o-alg. generated by open subsets.
  - Closed under complement, countable unions & intersections.
  - Contains \( \emptyset \) and \( G \).

A measure is \( \mu : \Sigma \to IR \cup \{0\} \) s.t.

a) \( \mu(E) \geq 0 \) \( \forall E \in \Sigma \).

b) \( \mu(\emptyset) = 0 \)

c) Countably additive, i.e.,

\[ \mu \left( \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n) \]

for \( E_1, E_2, \ldots \) pairwise disjoint.

**Thm (Haar)** There's a Borel measure \( \mu \) on \( G \):

i) Translation invariant: \( \mu(xE) = \mu(E) \) all \( x \in G \).

ii) \( \mu(E) < \infty \) for compact \( E \).

iii) Outer regular: \( \forall E \in \Sigma \),

\[ \mu(E) = \inf \left\{ \mu(U) : U \supseteq E \text{ open} \right\} \]

iv) Inner regular: \( \forall \text{ open } E \),

\[ \mu(E) = \sup \left\{ \mu(C) : C \subseteq E \text{ cpt.} \right\} \]

Such \( \mu \) is unique up to \( c > 0 \).
\( \mathbb{K} = \mathbb{K} \) \( \mu = \text{Lebesgue measure}, \ \mu(x + E) = \mu(E) \)

--- get \( \mu \) on our (non-arch.) \( \mathbb{K} \) \( \text{local field} \) \( \text{(add. group)} \)

Let \( x \leq K^\times \). Consider \( E \rightarrow \mu_x(E) \)

\( \Xi \sim \) another Haar measure.

By uniqueness, (translation - invt.?) \( \mu_x(a + E) = \mu(xa + xE) = \mu(xE) = \mu_x(E) \) ?

\( \mu_x(E) = \mu(x \cdot E) = c_x \mu(E), \ \forall E \in \Xi \).

**Claim:** \( c_x = |x|_K \) (= normalized abs. value.)

Why? 1st check i.e., \( \mu(xE) = |x|_K \mu(E) \).

\( |x| := c_x \) def. an abs. value on \( K \).

multiplicative \( \\checkmark \) strong triangle ineq.: Suppose \( \forall x \in K \), i.e. \( y \in xR \).

\( \|x + y\| \leq \mu(E) = \mu((x + y)E) \)

Take \( E = \mathbb{R} \) here \( \leq \mu(xE) = \|x\| \mu(E) \).

By regularity \( \mu(E) \neq 0 \). Thus

\( \|x + y\| \leq \|x\| = \max \{ \|x\|, \|y\| \} \)

--- take \( E = \mathbb{R} \) here

Thus \( y \in xR \) implies \( \mu(yR) \leq \mu(xR) \) i.e., \( \|y\| \leq \|x\| \).
REMAINS: \[ 11 \pi = q^{-1} \quad (i.e., \mu(\pi R) = q^{-1} \mu(R)) \]

Note: \( \pi R = m \), so \( R = \bigcup x + m \)

\[ \Rightarrow \mu(R) = \sum_{x \in S} \mu(x + m) \quad \text{representatives for \( k \)} \]

\[ = \sum_{x \in S} \mu(m) = q \mu(\pi R) \]

Remark:

\[ = \sum_{x \in S} \mu(m) = q \mu(\pi R) \]

For \( C \):

\[ \mu(x E) = |x|_2^2 \cdot \mu(E) \]

\( (E \subseteq B \text{ opt. ball say}) \quad \text{not an abs. value} \)

\( x \in E \text{ scale radius by } |x|_\infty = \text{modulus} \)
Newton's Method: \( K \) with non-arch. 1-1, complete, \( R \), \( \mathbb{D} \). 
\[ f(x) = a_0 + a_1 x + \ldots + a_n x^n \in R[x] \]  
(not nec. discrete) 

Suppose \( x_0 \in R \) sat. \( |f(x_0)| < |f'(x_0)|^2 \). 

Def. \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \) \( \quad (\forall n > 0) \) "Newton" 

Claim: Well-defined, i.e. all \( f'(x_n) \neq 0 \). 
- In fact, \( |f'(x_n)| = \ldots = |f'(x_0)|. \) 

More importantly: 

- \( (x_n) \) converges, and \( x := \lim_{n \to \infty} x_n \in R \) is a root of \( f \). 

\[ |x - x_0| \leq \left| \frac{f(x_0)}{f'(x_0)^2} \right| < 1. \] 

Why? Let \( c = \left| \frac{f(x_0)}{f'(x_0)^2} \right| < 1. \) 

- By induction we'll show: 
  (i) \( |x_n| < 1 \) 
  (ii) \( |x_n - x_0| < c \) 
  (iii) \( |f(x_n)| \leq c^{2n} |f'(x_n)|^2 \) 

\( (n = 0 \text{ ok}) \).
Assume $n \geq 0$ and ok for $n$. First, knowing $\frac{|f'(\alpha_n)|}{f'(\alpha_n)^2} \leq c^{2n}$ gives (by def. $\alpha_{n+1}$):

$$|\alpha_{n+1} - \alpha_n| \leq c^{2n}$$

- in particular this is $< 1$
and thus $|\alpha_{n+1}| \leq 1$; (i) (since $|\alpha_n| \leq 1$)

For (ii) note:

$$|\alpha_{n+1} - \alpha_0| \leq \max\{ |\alpha_{n+1} - \alpha_n|, |\alpha_n - \alpha_0| \} = c^{2n} \leq c$$

Finally (iii): Taylor,

$$f(\alpha_{n+1}) = f(\alpha_n) - f'(\alpha_n) \cdot \frac{f(\alpha_n)}{f'(\alpha_n)} + c^2 \left( \frac{f(\alpha_n)^2}{f'(\alpha_n)^2} \right) + \ldots$$

Thus, $\beta \in \mathbb{R}$,

$$|f(\alpha_{n+1})| \leq \left| \frac{f(\alpha_n)}{f'(\alpha_n)} \right|^2 =: e^2$$

Combine this with Taylor for $f'$:

$$f'(\alpha_{n+1}) = f'(\alpha_n) - f''(\alpha_n) \cdot \frac{f(\alpha_n)}{f'(\alpha_n)} + \ldots$$

abs $\geq \varepsilon$ abs $\leq \varepsilon$ abs $\leq \varepsilon^2$

Note: $\frac{f(\alpha_n)}{f'(\alpha_n)} = \beta \cdot \frac{f(\alpha_n)^2}{f'(\alpha_n)}$ for some $\beta \in \mathbb{R}$.

\[ R \Rightarrow \text{ shows } \varepsilon \leq c^{2n} \]

\[ \Rightarrow \text{ see next page.} \]
First term has \( |f'(c_{n})| \geq c^{-2n} \). \[
\left| \frac{f(c_{n})}{f'(c_{n})} \right| = \frac{c^{-2n}}{c^{-2n}} = c^{-2n} > \varepsilon. \]

Conclude (w. (ii)) that
\[
\left| \frac{f(c_{n+1})}{f'(c_{n+1})} \right| \leq \left| \frac{f(c_{n})}{f'(c_{n})} \right|^2 \cdot \frac{1}{|f'(c_{n})|^2} = \left| \frac{f(c_{n})}{f'(c_{n})} \right|^2 \leq c^{2n+1} \]

Shows (i) \( \Rightarrow \) (iii).

Back to claim: \( \square \) shows \( (c_{n}) \) is Cauchy.

By (iii), \( \alpha \) is definite integral \( \alpha = \lim_{n \to \infty} \alpha_{n} \) (K complete.)

\( f(n) \to f(\alpha) = 0. \) root.

At least, check \( |\alpha_{n} - \alpha| \leq c \) in (ii). Let \( n \to \infty \).

Summary: K complete (non-arch.)

\( f \in \mathbb{R}[x] \). Suppose \( \alpha_{0} \in \mathbb{R} \) s.t. \( |f(\alpha_{0})| < |f'(\alpha_{0})| \).

Then \( \exists \alpha_{1} \in \mathbb{R} \) s.t. \( (1) f(\alpha) = 0 \)

\( (2) |\alpha - \alpha_{0}| \leq \left| \frac{f(\alpha_{0})}{f'(\alpha_{0})} \right| < 1. \)