WEEK 6.

<u>General definition</u>: E/K finite, w. compatible $|\cdot|$. (non-arch.)

- We say E/K is "<u>unramified</u>" if
  - k_E / k_K separable, and
  - $f(E/K) = [\hat{E} : \hat{K}]$.

- When K complete, $|K^\times|$ discrete, k_K <u>perfect</u>, (ex. <u>local field</u>) this just means:

$$\boxed{e(E/K) = 1}. \qquad (e \cdot f = [E:K])$$

[Assume from now on we're in this situation.] <u>Setup</u>:

$C = \overline{K}$ with the extension of $|\cdot|$.

$k_C = \mathcal{O}_C / m_C$ is an alg. <u>closure</u> of k_K.

(why? • <u>algebraic</u>: Consider any $\overline{\alpha} \in k_C$.

$f(t) = \mathrm{Irr}(\alpha, K, t)$ has <u>coeffs</u>. in $\mathcal{O}_K$ since $|\alpha| \leq 1$, as noted earlier. "Hensel–Kurschak"

- then $\overline{\alpha}$ root of $\overline{f}(t)$.

• <u>alg. closed</u>: Consider any monic $\overline{f} \in k_C[t]$.

Since C alg. closed,

$$f(t) = \prod (t - \alpha_i), \quad \alpha_i \in C$$

integral so in $\mathcal{O}_C$

reduce:

$$\overline{f}(t) = \prod (t - \overline{\alpha}_i)$$

splits ✓)

★ Convention: All alg. "extensions" of K resp. k_K will be subfields of C resp. k_C.]

$$\begin{array}{c} C \\ | \\ E \\ \text{alg.}\ | \\ K \end{array}$$

- Natural map:

$$\left\{\begin{array}{c}\text{finite}\\ \text{extns.}\\ E/K\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{finite}\\ \text{extns.}\\ e/k\end{array}\right\}$$

$$E \longmapsto k_E.$$

sat.: $[E:K] \geq [e:k]$.

- map is surjective. More precisely, given e/k there's an E/K s.t. $\begin{cases} \circ\ e = k_E \text{ and} \\ \circ\ [E:K] = [e:k]. \\ \quad\text{(unram.)} \end{cases}$

Why? e/k sep., so $e = k(\bar{\alpha})$ for some $\alpha \in \mathcal{O}_C$. "primitive elt."

Consider $\mathrm{Irr}(\bar{\alpha}, k, t)$, and pick monic lift $f \in \mathcal{O}_K[t]$.

Factor $f(t) = \prod (t - \alpha_i)$ with $\alpha_i \in \mathcal{O}_C$.

Have $\bar{\alpha} = \bar{\alpha_i}$ for some (unique) i.

CLAIM: $E = K(\alpha_i)$ works.

$\mathcal{O}_K[\alpha_i] \subset \mathcal{O}_E = E \cap \mathcal{O}_C$ so

at least $\quad e = k(\bar{\alpha}) \subset k_E$.

$\uparrow$ equal:

$$[k_E : k_K] \leq [E:K] \overset{=}{\leq} \deg(f) = \deg(\bar{f}) = [e:k] \quad \checkmark$$

- <u>NOTE</u>: The "preimage" of e/k constructed above is <u>unramified</u>.

<u>Thm</u>. (Same setup.) There's a <u>bijection</u>,

$$\left\{ \begin{matrix} \text{finite } \underline{\text{unramified}} \\ \text{extns. } E/K \end{matrix} \right\} \xrightarrow{1:1} \left\{ \begin{matrix} \text{finite} \\ \text{extns. } e/k \end{matrix} \right\}$$

(degree-preserving)

- Thus, if k finite (i.e. K <u>local</u>) $\forall n$ there's a <u>unique</u> <u>unramified</u> extn. of degree n.

<u>Recipe</u>: Say $|k| = q$. Its degree n extn. is $e = k(\bar{\zeta})$ where $\zeta \in C$ primitive $(q^n - 1)$st root of unity.

$E = K(\zeta)$.

(separable)

Galois $\leftarrow$ splitting field of $X^{q^n-1} - 1$.

- <u>Notation</u> $(K = \mathbb{Q}_p)$

$$\begin{array}{c} \mathbb{Q}_{p^n} \\ n \Big| \\ \mathbb{Q}_p \end{array}$$

Injective? Stronger statement:

- Recall: "recipe" If e/k finite, say $e = k(\bar{\alpha})$. (choices)

$\mathrm{Irr}(\bar{\alpha}, k, t) = \bar{f}(t)$, $f \in \mathcal{O}_K[t]$ monic lift.

Since $\mathcal{O}_C$ is (strictly) Henselian, there's a unique lift $\alpha \in \mathcal{O}_C$: $f(\alpha) = 0$.

Then $K(\alpha) = E$ is an unramified$/K$ with $k_E = e$.

CLAIM:

- If F/K is any finite extn. with $k_F = e$, then F contains the above unramified E: $\boxed{E \subseteq F}$.

[Thus, if F is also unramified:

$$[F:K] = [k_F : k] = [e:k] = [k_E : k] = [E:K],$$

so that $E = F$.]

Why? View $f \in \mathcal{O}_F[t]$, $\bar{f} \in k_F[t]$ has simple root $\bar{\alpha} \in k_F$ $(= e)$

Since $\mathcal{O}_F$ Henselian, $\exists! \beta \in \mathcal{O}_F$:
- $\bar{\beta} = \bar{\alpha}$
- $f(\beta) = 0$.

"Uniqueness of lifts" ($\bar{\alpha}$ simple) gives $\beta = \alpha$

Therefore $E = K(\alpha) \subseteq F$. $\square$

- Assume $K, |\cdot|$ is non-arch.

  a local field (so complete, discretely valued, k finite)

$p = \mathrm{char}(k)$

$q = |k|$

THE deg. n unramified extn.:

$$K(\zeta), \quad \mathrm{ord}(\zeta) = q^n - 1.$$

- More generally:

  - Exc For $p \nmid m$, $K(\zeta_m)$ unramified of degree $\mathrm{ord}_{(\mathbb{Z}/m\mathbb{Z})^\times}(q)$.

So, in particular:

$$K^{ur} = \bigcup_{\substack{E/K \\ \text{fin. unram.}}} E = \bigcup_{p \nmid \mathrm{ord}(\zeta)} K(\zeta)$$

"adjoin all roots of unity of order prime-to-p".

- Another consequence (useful later):

$$\begin{array}{c} K^{ur} \\ | \\ K \end{array}$$

$$\mu_{p'}(K) := \left\{ \begin{array}{l} \text{roots of unity in } K \\ \text{of order prime-to-}p \end{array} \right\}$$

$$= \mu_{q-1}(K) \xrightarrow[\text{Teichüller}]{\sim} k^\times.$$

($\supseteq$ obvious) $\subseteq$: Say $\zeta_m \in K$ where $p \nmid m$.

By exc., $\mathrm{ord}_m(q) = [K:K] = 1$.

I.e., $m \mid (q-1)$. Thus $\zeta_m^{q-1} = 1$. ✓

Galois groups? E/K finite.

Natural homomorphism $\mathrm{Gal}(E/K) \longrightarrow \mathrm{Gal}(k_E/k_K)$.

$$\gamma \longmapsto \overline{\gamma}$$

(How? $E \ni x \mapsto |\gamma(x)|$ ext. $|\cdot|$ on K
$= |x|$ — uniqueness; K complete.)

Thus $\gamma : \mathcal{O}_E \xrightarrow{\sim} \mathcal{O}_E$ and $\gamma : \mathfrak{m}_E \xrightarrow{\sim} \mathfrak{m}_E$.

On residue fields $\overline{\gamma} : k_E \xrightarrow{\sim} k_E$, k_K-linear

Thm If E/K unramified, this is an isomorphism.

$$\mathrm{Gal}(E/K) \xrightarrow{\sim} \mathrm{Gal}(k_E/k_K)$$

Pf. Both have size $[E:K] = [k_E : k_K]$, so suff. to show injective:

$$\overline{\gamma} = \mathrm{Id} \overset{?}{\Longrightarrow} \gamma = \mathrm{Id}.$$

Recall $E = K(\alpha)$ where $\alpha \in \mathcal{O}_E$ is a lift of $\overline{\alpha}$, and $k_E = k_K(\overline{\alpha})$.

$\mathrm{Irr}(\overline{\alpha}, k, t) = \overline{f(t)}$

$f(\alpha) = 0$.

$\gamma(\alpha) \in E$ is (another) root of f, reducing to $\overline{\gamma}(\overline{\alpha}) \underset{\overline{\gamma} = \mathrm{Id}}{=} \overline{\alpha}$.

Must have $\gamma(\alpha) = \alpha$, otherwise $\mathrm{Irr}(\overline{\alpha}, k, t)$ would have multiple roots $\overline{\alpha}$. $\square$

- Moreover, $\mathrm{Gal}(k_E/k) = \langle x \mapsto x^q \rangle$ cyclic.

Thus $\mathrm{Gal}(E/K) = \langle \varphi_{E/K} \rangle$

"Frobenius automorphism"

$$\varphi_{E/K}(x) \equiv x^q \pmod{m_E}.$$
$$x \in O_E$$

Writing $E = K(\zeta)$,

$\mathrm{ord}(\zeta) = q^n - 1$,

$\varphi_{E/K}(\zeta) = \zeta^q$ (both sides reduce to the same root of 1 in k_E, $\mu_{q^n-1}(E) \xrightarrow{\sim} k_E^\times$)

- Works for K^{ur}.

$$\mathrm{Gal}(K^{ur}/K) \xrightarrow{\sim} \mathrm{Gal}(\bar{k}/k)$$

topologically cyclic $\simeq \hat{\mathbb{Z}}$.

$$\mathrm{Gal}(K^{ur}/K) = \overline{\langle \varphi \rangle}$$

abelian!

Def. The "inertia group" of K is the (closed) subgroup:

$$I_K = \mathrm{Gal}(\bar{K}/K^{ur}).$$

$$0 \to I_K \longrightarrow \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow 0$$

"Local class field theory": Understand $K^{ab} \supseteq K^{ur}$.

Def. The "Weil group" of K is a dense subgroup of $\mathrm{Gal}(\overline{K}/K)$:

$$W_K = \{\gamma : \overline{\gamma} \text{ is a } \mathbb{Z}\text{-power of Frobenius}\}.$$

$$\begin{array}{ccccccccc}
0 & \to & I_K & \longrightarrow & W_K & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \cap & & \cap & & \\
0 & \to & I_K & \longrightarrow & \mathrm{Gal}(\overline{K}/K) & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 0
\end{array}$$

Topology on W_K not induced from $\mathrm{Gal}(\overline{K}/K)$.

Insist I_K is open.

Thm (LCFT) $\quad K^\times \xrightarrow{\sim} W_K^{ab}$.

non-arch. π = uniformizer.

K local field. (complete, disc. valued, k finite)

- A finite E/K is "**totally ramified**" if $e(E/K) = [E:K]$

(i.e. $f(E/K) = 1$)

Ex. $f(t) = a_0 + a_1 t + \dots + a_{n-1} t^{n-1} + t^n$

$\in \mathcal{O}_K[t]$ **Eisenstein polynomial**, i.e.

- all $v_K(a_i) \geq 1$.
- $v_K(a_0) = 1$.

($\Rightarrow$ irreducible). I.o.w., $\pi \mid a_i$ for all i but $\pi^2 \nmid a_0$.

$$\forall \begin{array}{c} E \\ \mid \text{tot. ram.} \\ F \\ \mid \text{unram.} \end{array}$$

- Here: K

($k_E = k_F$)

Pick $\Pi \in \mathbb{C} = \overline{K}$: $f(\Pi) = 0$.

- Then $K(\Pi)/K$ is **tot. ram.**: (w. uniformizer Π)

First obs. $|\Pi| < 1$:

- indeed, Π **integral**$/\mathcal{O}_K$ (prev. obs.: $|\Pi^n| \leq \dots$)

- **Second**, thus $\pi \mid \Pi^n$

~~$|\Pi| =$~~

$$|\pi| = |a_0| > |a_i \Pi^i| \quad \text{for } i > 0.$$

- shows $\max\{|a_0|, |a_1\Pi|, \dots, |a_{n-1}\Pi^{n-1}|\}$ achieved **once**

$$\| $$

$$|\Pi^n| = |\pi|$$

- In particular, $\underbrace{1 > |\Pi| > |\Pi^2| > \cdots > |\Pi^{n-1}|}_{\text{all } \underline{\text{distinct}} \text{ mod } |K^\times| = |\pi|^{\mathbb{Z}}} > |\Pi^n| = |\pi|$

$\Rightarrow e(K(\Pi)/K) \geq n$.

On the other hand $e(K(\Pi)/K) \leq [K(\Pi):K] \underset{f \text{ irr.} \checkmark}{=} n$.

Π uniformizer: $|K(\Pi)|^\times = |\Pi|^{\mathbb{Z}}$.

[__Thm__. If E/K tot. ram., $E = K(\Pi)$ for __any__ choice of uniformizer $\Pi \in E$, and $\mathrm{Irr}(\Pi, K, t)$ is Eisenstein.

PROOF. Pick unif. $\Pi \in E$. __Claim__ $K(\Pi) = E$:

$$[K(\Pi):K] \geq e(K(\Pi)/K) = e(E/K) \xlongequal{\text{tot. ram.}} [E:K].$$

$$\begin{array}{c} E \\ | \\ K(\Pi) \\ | \\ K \end{array}$$

why?
Same valuation grp.:

$$|E^\times| = |\Pi|^{\mathbb{Z}} \underset{\text{(equality)}}{\leq} |K(\Pi)^\times| \quad \checkmark$$

$f(t) = \mathrm{Irr}(\Pi, K, t)$

__Eisenstein__:

Factor $f(t) = \prod_{i=1}^{n} (t - \alpha_i)$, $\alpha_i \in C$. $(\alpha_1 = \Pi)$

__Claim__: $|\alpha_i| = |\Pi| \quad \forall i$.

Extend $E \cong K[t]/(f) \longrightarrow C$ to $\gamma \in \mathrm{Gal}(C/K)$

$\Pi \longleftarrow t + (f) \longmapsto \alpha_i \qquad \gamma: \Pi \longmapsto \alpha_i$

Then $|\alpha_i| = |\gamma(\Pi)| = |\Pi|$

— Now $\left.\begin{array}{l} a_{n-1} = \pm(\alpha_1 + \cdots + \alpha_n) \\ \vdots \\ a_0 = \pm(\alpha_1 \cdots \alpha_n) \end{array}\right\}$ all have $|a_i| < 1$ and $|a_0| = |\Pi|^n \underset{\substack{\uparrow \\ E/K \text{ tot. ram.}}}{=} |\pi|$

$\square$

Ex. ($K = \mathbb{Q}_p$)

ζ = primitive p^{th} root of 1 in $\overline{\mathbb{Q}_p}$

$E = \mathbb{Q}_p(\zeta)$, totally ramified $/\mathbb{Q}_p$: $\mathrm{Irr}(\zeta - 1, \mathbb{Q}_p, t) = \Phi_p(t+1)$ Eisenstein.

w. uniformizer $\zeta - 1$.

$= \dfrac{(t+1)^p - 1}{(t+1) - 1} = \cdots$

[direct proof:

$$p = \Phi_p(1) = \prod_{i=1}^{p-1}(1 - \zeta^i) = \underbrace{\left(\prod_{i=1}^{p-1} \frac{1-\zeta^i}{1-\zeta}\right)}_{\substack{\Uparrow \\ \mathbb{Z}_p[\zeta]^\times}}(1-\zeta)^{p-1} \sim (1-\zeta)^{p-1} \text{ .]}$$

NOTE: $\begin{array}{c} \mathbb{Q}_p(\zeta) \\ | \\ \mathbb{Q}_p \end{array}$ KUMMER extension (meaning

- $\mu_{p-1} \leqslant \mathbb{Q}_p^\times$
- $\mathrm{Gal} \simeq (\mathbb{Z}/p\mathbb{Z})^\times$ killed by $p-1$.

$$\boxed{\begin{array}{l} \underline{\text{Def}} \quad E/K \text{ ``Kummer''} \\ \text{(of exponent} \mid n\text{) if} \\ \circ\ \mu_n \leqslant K^\times \\ \circ\ \mathrm{Gal}(E/K) \text{ abelian,} \\ \quad \text{killed by } n, \end{array}}$$

$$\mathbb{Q}_p(\zeta) = \mathbb{Q}_p(\sqrt[p-1]{-p})$$

– Why? By "direct proof",

$$p = \underbrace{\prod_{i=1}^{p-1} (1+\zeta+\dots+\zeta^{i-1})}_{\text{unit } u \equiv \prod_{i=1}^{p-1} i \equiv -1 \bmod (1-\zeta)} \cdot (1-\zeta)^{p-1}$$

Consider $f(t) = t^{p-1} + u \in \mathbb{Z}_p[\zeta][t]$.

$\bar{f}(t) = t^{p-1} - 1 \in \mathbb{F}_p[t]$. – reduce mod $(1-\zeta)$.

– pick any root and lift (Hensel) it to $c \in \mathbb{Z}_p[\zeta]$: $c^{p-1} + u = 0$.

Then, $-p = (-u) \cdot (1-\zeta)^{p-1} = (c(1-\zeta))^{p-1}$

shows $\mathbb{Q}_p(\sqrt[p-1]{-p}) \subseteq \mathbb{Q}_p(\zeta)$.

same degree $p-1$ so $=$.

Def. E/K fin. extn. of local fields. $p =$ residual characteristic. $= \mathrm{char}(k)$

E/K is "tamely ramified" if $p \nmid e(E/K)$.

[othw. wildly ram.].

$$\begin{tikzcd} E \arrow[d, dash, "\text{wild}"] \\ M \arrow[d, dash, "\text{tame}"] \\ F \arrow[d, dash, "\text{unr.}"] \\ K \end{tikzcd}$$

$\forall$ tot.

Lemma If E/K is totally & tamely ram., fix uniformizer $\pi \in K$.

(1) E admits a uniformizer Π s.t. $\Pi^e = \zeta\pi$ for some $\zeta \in \mu_{q-1}(K)$. (and $E = K(\Pi)$)

(2) If E/K is Galois, $q \equiv 1 \pmod{e}$, and $\mathrm{Gal}(E/K)$ cyclic of order e. (so a "Kummer extn." of exp. $|e$)

PROOF. Pick uniformizers $\pi \in K$, $\Pi \in E$. Write $\pi = u\Pi^e$. (u: unit $\in \mathcal{O}_E^\times$; put u on this side.)

Note: $k_E = k$ (tot. ram.) $\quad q = |k|$

Thus,
$$\mu_{q-1}(K) \xrightarrow[\text{mod } m_K]{\sim} k^\times = k_E^\times$$

$\zeta =$ Teichmüller lift of $\bar{u}$. $\in \mu_{q-1}(K)$

$\zeta \mapsto \bar{u}$

cf. Teichmüller

$$\boxed{\begin{aligned} \mathcal{O}_E^\times &\simeq k_E^\times \times (1+m_E) \\ u &\mapsto (\bar{u}, u_1) \end{aligned}}$$

Then $u = \zeta v$ for some $v \in 1+m_E$.

Consider $f(t) = t^e - v \in \mathcal{O}_E[t]$. By Hensel,

$\begin{cases} \circ\ f(1) \equiv 0\ (\mathfrak{m}_E) \\ \circ\ f'(1) = e \not\equiv 0\ (\mathfrak{m}_E) \end{cases}$ – since $p \nmid e$ (tame ram.)

$\exists\, c \in \mathcal{O}_E$ s.t. $c \equiv 1\ (\mathfrak{m}_E)$ and $c^e - v = 0$.

– Modify Π. Look at $\tilde{\Pi} := c^{-1}\Pi$ (uniformizer ✓) satisfies $\tilde{\Pi}^e = \zeta\pi$

(check: $\tilde{\Pi}^e = c^{-e}\Pi^e = v^{-1}u\pi = \zeta\pi$)
using $u\pi = \Pi^e$

– Modify π. Look at $\tilde{\pi} := \zeta\pi$ (uniformizer ✓)

Since E/K tot. ram., know $E = K(\tilde{\Pi})$ (any uniformizer).

& $\tilde{\Pi}$ is an e^{th} root of $\tilde{\pi}$.

NOW suppose E/K is Galois.

$$X^e - \tilde{\pi} = \prod_{i=1}^{e} (X - \alpha_i) \quad \text{w. } \alpha_i \in E \text{ distinct.}$$

(Galois)

$X^e - \tilde{\pi}$ irreducible in $K[X]$ by Eisenstein. ($= \mathrm{Irr}(\tilde{\Pi}, K, X)$)

– has at least one root in E, namely $\tilde{\Pi}$.

Here $\alpha_i = \tilde{\pi} \cdot (e^{th}$ root of 1 in $E)$.

- Since we have e distinct roots $\alpha_1, \dots, \alpha_e$, must have e distinct e^{th} roots of 1 in E:

$$|\mu_e(E)| = e.$$

Thus,

$$\mu_e(E) \leqslant \mu_{p'}(E) \overset{\text{prev. obs.}}{=\!=} \mu_{q-1}(E) = \mu_{q-1}(K)$$

$p \nmid e$

equivalently, $e \mid (q-1)$.

Conclude E ($=$ splitting field of $X^e - \tilde{\pi}$) is a <u>Kummer</u> <u>extension</u> of exponent (div.) e.

- Why is $\mathrm{Gal}(E/K)$ <u>cyclic</u>? Inj. hom.,

$$\begin{aligned} \mathrm{Gal}(E/K) &\xrightarrow{\sim} \mu_e(E) \\ \sigma &\longmapsto \frac{\sigma(\tilde{\pi})}{\tilde{\pi}} \end{aligned} \qquad \square$$

$$\begin{array}{c} E \\ e \big| \\ K \end{array}$$

- $\mu_e \leqslant K^\times$ abelian
- $\mathrm{Gal}(E/K)$ killed by e.

<u>Ex.</u> (cont.) $\mathbb{Q}_p(\zeta) = \mathbb{Q}_p(\sqrt[p-1]{-p})$.

$(\zeta^p = 1)$

$$\begin{array}{c} \mathbb{Q}_p(\zeta) \\ \big| \\ \mathbb{Q} \end{array}$$

tot. & tamely ram.
$(e = p-1)$

E/K totally ram. extn. (of local fields). $\quad p = \mathrm{char}(k)$.

Assume E/K Galois, $G = \mathrm{Gal}(E/K)$.

$H =$ some Sylow p-subgroup.

$$\begin{array}{l} E \\ H \Big| \text{ p-power degree} \\ M = E^H \\ G/H \Big| \text{ (prime-to-p degree)} \\ K \end{array}$$

~ Here M/K totally & tamely ram., so $\exists$ uniformizer $\Pi \in M$ s.t.

$$\Pi^e = \underbrace{\zeta}_{\substack{\text{some root of 1} \\ \text{in } \mu_{q-1}(K)}} \underbrace{\pi}_{\substack{\forall \text{ predetermined} \\ \text{uniformizer for } K}} = \tilde{\pi}, \quad e = [M:K].$$

CLAIM: M/K Galois.

(i.e., $H \triangleleft G$ is THE Sylow pro-p-subgroup).

$\mathrm{Irr}(\Pi, K, x) =$

$$\underbrace{x^e - \tilde{\pi}}_{\substack{\text{irred.} \\ \Uparrow \\ K[x]}} = \prod_{i=1}^{e}(x - \alpha_i)$$

~ has some root in E (namely Π)

so all $\alpha_i \in E$ (E/K Galois).

$\alpha_i = \Pi \cdot (e^{th}$ root of 1 in $E)$

Note:

$$\mu_e(E) \leq \mu_{p'}(E) = \mu_{q-1}(E) = \mu_{q-1}(K) \leq K^\times$$

$p \nmid e$ $\qquad q = |k_E| = |k|$ (tot. ram.)

so all $\alpha_i \in M$.

~ Therefore $M = K(\Pi)$ is Galois $/K$.

[As we've seen, Gal(M/K) is <u>cyclic</u> of order e.]

In general, E/K finite Galois.

$G = \mathrm{Gal}(E/K)$

$I = \mathrm{Gal}(E/\underset{F}{M}) \lhd G$ "<u>inertia</u>".

$P = \mathrm{Gal}(E/M) \lhd I$ the Sylow p-subgroup. "<u>wild</u> <u>inertia</u>".

I/P "tame inertia" $(\cong \mathrm{Gal}(M/F))$ — already <u>obs</u>: cyclic of order prime-to-p.

$$\begin{array}{lll} & E & \\ & \mid \text{ wild.} & P \\ \text{tot. ram.} & M & \\ & \mid \text{ tame} & I/P \\ & F & \\ & \mid \text{ unr.} & G/I = \langle \varphi \rangle \\ & K & \end{array}$$

<u>Remark</u>:
p-groups are solvable (center $\neq \{1\}$)
so G is <u>solvable</u>.

$\forall n > 0$.

Pick a root $\pi_n \in \overline{\mathbb{Q}_p}$ of $\underbrace{X^{p^n-1} + p}_{\text{Eisenstein in } \mathbb{Z}_{p^n}[X]} = 0$.

(unique $/ \mu_{p^n-1}$).

$\mathbb{Q}_{p^n}(\pi_n)$ ∘ independent of choice π_n

∘ degree $p^n - 1$, Galois extn.

∘ totally (and tamely) ramified.

$\mathbb{Q}_{p^n}$

∘ $\mathbb{Q}_p(\pi_1) = \mathbb{Q}_p(\zeta_p)$.

Note: $\mathrm{Gal}(\mathbb{Q}_{p^n}(\pi_n)/\mathbb{Q}_{p^n}) \xrightarrow{\sim} \mu_{p^n-1}(\overline{\mathbb{Q}}_p) \xrightarrow{\sim} \mathbb{F}_{p^n}^\times$,

$\sigma \longmapsto \dfrac{\sigma(\pi_n)}{\pi_n}$

cyclic (prime-to-p order).

Exc "max tamely ramn.":

$$\mathbb{Q}_p^{tr} := \bigcup_{\substack{E/\mathbb{Q}_p \\ \text{finite,} \\ \text{tamely ramified}}} E \overset{\downarrow}{=} \bigcup_{n>0} \mathbb{Q}_{p^n}(\pi_n).$$

Hint: Supp. $E/\mathbb{Q}_p$ tame, $e = e(E/\mathbb{Q}_p)$.

Know $E = \mathbb{Q}_{p^m}(\Pi)$ for some uniformizer $\Pi \in E$ s.t. $\Pi^e = \zeta(-p)$ for some $\zeta \in \mu_{p^m-1}$. Write $\zeta = \xi^e$ for some $\xi \in \mu_{e(p^m-1)}$. Then $\Pi = \underset{\mu_e}{\lambda} \cdot \xi \cdot \pi_n^{(p^n-1)/e}$ for any $n > 0$ s.t. $p^n \equiv 1 \pmod{e}$.

lies in $\mathbb{Q}_{p^n}(\pi_n)$ for $n \gg 0$.

$$\begin{array}{c} E \\ e \,\big|\, \text{tot \& tame} \\ \mathbb{Q}_{p^m} \\ | \\ \mathbb{Q}_p \end{array}$$

Check: For $m|n$, may take $\pi_m = \pi_n^{(p^n-1)/(p^m-1)}$

$$
\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}_{p^n}(\pi_n)/\mathbb{Q}_{p^n}) \arrow[r, "\sim"] \arrow[d, "\mathrm{res.}"] & \mathbb{F}_{p^n}^\times \arrow[d, "\mathrm{Norm.}"] \\
\mathrm{Gal}(\mathbb{Q}_{p^m}(\pi_m)/\mathbb{Q}_{p^m}) \arrow[r, "\sim"] & \mathbb{F}_{p^m}^\times
\end{tikzcd}
$$

"Tame inertia":

$$I/P = \mathrm{Gal}(\mathbb{Q}_p^{tr}/\mathbb{Q}_p^{ur}) \xrightarrow{\sim} \varprojlim \mathrm{Gal}(\mathbb{Q}_{p^n}(\pi_n)/\mathbb{Q}_{p^n})$$

$$\xrightarrow{\sim} \varprojlim \mathbb{F}_{p^n}^\times$$

(pro-cyclic).

$P = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{tr})$
the Sylow pro-p.