WEEK
7.

K <u>local field</u> (admits $|\cdot|$ s.t. K locally compact). complete.

If archimedean, know $K = \mathbb{R}$ or $K = \mathbb{C}$
(Ostrowski 2nd)

Assume <u>non</u>-archimedean. Know amounts to: complete, discretely-valued, k finite.

<u>Claim</u>: 1) If $\mathrm{char}(K) = 0$,
$K/\mathbb{Q}_p$ finite.

2) If $\mathrm{char}(K) = p$,
$K = \mathbb{F}((t))$ for some finite $\mathbb{F} \supset \mathbb{F}_p$.
(equiv. K fin. extn. of $\mathbb{F}_p((t))$.)

Why? 1) Supp. $\mathbb{Q} \subseteq K$. Let $p = \mathrm{char}(k)$, $k = \mathcal{O}_K/\mathfrak{m}_K$.

Note $|p| < 1$ (since $p \in \mathfrak{m}_K$).

(- replace $|\cdot|$ by $|\cdot|^c$ for suitable $c > 0$ to arrange
$|p| = p^{-1}$)

Then $|\cdot|_{\mathbb{Q}} = |\cdot|_p$ since $p^n \to 0$. (Ostrowski for $\mathbb{Q}$)

K complete, so $\mathbb{Q}_p \subseteq K$. Finite?

complete & disc.-val.

$\downarrow$

$$[K:\mathbb{Q}_p] = e(K/\mathbb{Q}_p) \cdot f(K/\mathbb{Q}_p)$$
$$= v_K(p) \cdot [k:\mathbb{F}_p] < \infty.$$

2) Supp. $\mathrm{char}(K) = p$. Note that also $\mathrm{char}(k) = p$.
$\mathbb{F}_p \subseteq K$. $(\underbrace{1+\dots+1}_{p} = 0$ in $\mathcal{O}_K$; reduce mod $m_K)$

Recall:

$\mu_{q-1}(K) \xrightarrow{\sim} k^\times$, $q = |k|$.

$\leadsto$ pick $\zeta \in \mu_{q-1}(K)$ primitive. Consider $\mathbb{F}_p(\zeta) \subseteq K$. $\mathbb{F}_p(\zeta) =: \mathbb{F}$

Select uniformizer $t \in K$.
Then $\forall a \in K$ has a Laurent expansion

(actually $\mathbb{F} \subseteq \mathcal{O}_K$ and red.: $\mathbb{F} \xrightarrow{\sim} k$)

$a = \sum_{i=-N}^{\infty} a_i t^i$ w. all $a_i \in \mathbb{F}$.

representatives "S" (closed under $+, \cdot$)

I.e., $K \simeq \mathbb{F}((t))$. ✓

Global fields are
- finite ext. $K/\mathbb{Q}$ ("number fields")
- finite ext. of $\mathbb{F}_p(t)$ ("function fields")

Thm All local fields are completions of global fields, and vice versa.

obvious: All non-arch. $|\cdot|$ on $\mathbb{Q}$, $\mathbb{F}_p(t)$ are discrete w. finite res. field.
$(\mathbb{R}, \mathbb{C})$

Also, $\mathbb{F}((t))$ is completion of $\mathbb{F}(t)$. Substance:

$\forall$ $K/\mathbb{Q}_p$ finite is the completion of a number field.

<u>Strategy</u>: Produce dense subfield $F \subseteq K$
with $[F:\mathbb{Q}] = [K:\mathbb{Q}_p]$.

- Key input:

<u>KRASNER's LEMMA</u>: $K, |\cdot|$ complete (non-arch.)
$C = \overline{K}$ with extn. $|\cdot|$.

Supp. $\alpha \in C$ has a <u>separable</u> min. poly.:

$$\mathrm{Irr}(\alpha, K, t) = \prod (t - \alpha_i), \quad \alpha_1 = \alpha.$$

Supp. $\beta \in C$ satisfies the inequality

$$|\beta - \alpha| < |\alpha_i - \alpha|, \quad \forall i \neq 1.$$

[β is a better approximation to α than any of its conjugates].

Then, $K(\alpha) \subseteq K(\beta)$.

PROOF. Otherwise $K(\alpha,\beta) \neq K(\beta)$.

$\mathrm{Hom}_{K(\beta)}(K(\alpha,\beta), C)$ has a <u>non</u>-identity embedding.
(uses $K(\alpha)/K$ separable)

Say,

$$\sigma : K(\alpha,\beta) \to C, \quad K(\beta)\text{-linear}.$$

May extend to $\sigma \in \mathrm{Gal}(\mathbb{C}/K(\beta))$.

Let $\alpha_i := \sigma(\alpha)$; here $i \neq 1$.

Then,

$$\begin{aligned} |\beta - \alpha| &= |\sigma(\beta - \alpha)| \\ &= |\beta - \alpha_i| \\ &= \max\{|\beta - \alpha|, \underset{\text{bigger.}}{|\alpha - \alpha_i|}\} \\ &= |\alpha - \alpha_i| \end{aligned}$$

contradiction. $\square$

$K, |\cdot|$ complete.

Prop. $f \in K[t]$ irreducible, separable, monic. Then $\exists\, \varepsilon > 0$ with the property:

(A) | Any monic $g \in K[t]$ w. $\deg(g) = \deg(f)$ and $\|f - g\|_{max} < \varepsilon$ is irreducible.

PROOF. Factor $f(t) = \prod_{i=1}^{n} (t - \alpha_i)$, $\alpha_i \in C$ distinct.

- Step 1: Any $\beta \in C$ with $g(\beta) = 0$ satisfies $|\beta| \leq \|g\|$. (uses only g is monic)

$$g(t) = c_0 + c_1 t + \dots + c_{n-1} t^{n-1} + t^n.$$

If $|\beta| > \|g\|$, i.e. $|\beta| > |c_i|$ for all i,

$$|\beta^n| \leq \max\{ |c_0|, |c_1 \beta|, \dots, |c_{n-1} \beta^{n-1}| \} < |\beta|^n.$$

(used that g is monic so $\|g\| \geq 1$, thus $|\beta| > 1$)

- Step 2: $\|f\| = \|g\|$ for g as in (A) with $\varepsilon \leq 1$.

$$\|g\| = \max\{ \underbrace{\|g - f\|}_{< \varepsilon \text{ by assumption}}, \underbrace{\|f\|}_{f \text{ monic so } \geq 1} \} = \|f\|$$

- Step 3: If $g(\beta) = 0$ then $|f(\beta)| < \varepsilon \|f\|^n$

$f(t) = a_0 + \cdots + a_{n-1}t^{n-1} + t^n$.

Then, $|f(\beta)| = |f(\beta) - g(\beta)| \overset{\text{same deg.}}{=} \left|\sum_{i=1}^{n} (a_i - c_i)\beta^i\right| \leq$

$$\max_i |a_i - c_i| \cdot |\beta|^i \underset{\text{Step 1}}{\leq} \|f - g\| \cdot \|g\|^n < \varepsilon\|g^n\| \overset{\text{Step 2}}{=} \varepsilon\|f\|^n$$

—— in <u>other</u> words: $\forall$ root β of g,

$$\left|\prod_{i=1}^{n} (\beta - \alpha_i)\right| < \varepsilon\|f\|^n.$$

$\Rightarrow$ There must be <u>some</u> i s.t. $|\beta - \alpha_i| < \varepsilon^{1/n} \cdot \|f\|$.
(dep. only on f)

Taking ε <u>small</u> <u>enough</u> can <u>arrange</u>

$$\varepsilon^{1/n}\|f\| < |\alpha_j - \alpha_i| \quad \text{for all } j \neq i.$$

Since $f(t) = \mathrm{Irr}(\alpha_i, K, t)$, <u>KRASNER's</u> Lemma tells us that:

$$K(\alpha_i) \subseteq K(\beta)$$

degree $= n$ $\qquad$ degree $\leq \deg(g) = n$.

Must be equality. I.e., g <u>irreducible</u>. $\square$

Remark: Any pair f, g as in the Prop., has the "root exchange property":

$$\forall \text{root } \alpha \text{ of } f, \ \exists \text{ root } \beta \text{ of } g: \quad K(\alpha) = K(\beta).$$

(adopt notation from prev. proof): $\tau: K(\alpha_i) \xrightarrow{\sim} K(\alpha)$, $\alpha_i \longmapsto \alpha$

$\alpha = \alpha_j$ some j.

$\alpha_i =$ THE chosen root...

Found $K(\alpha_i) = K(\beta)$, apply τ:

$$K(\alpha) = K(\tau(\beta))$$

$\tau(\beta)$ ← root of $g(t)$. ✓

1. Application: $K/\mathbb{Q}_p$ finite. Find dense subfield $F \subseteq K$ with $[F:\mathbb{Q}] = [K:\mathbb{Q}_p] < \infty$. (showing $K = \widehat{F}$)

$K = \mathbb{Q}_p(\alpha)$

$f(t) = \operatorname{Irr}(\alpha, \mathbb{Q}_p, t)$

By Prop. $\exists\, \varepsilon > 0$ s.t. any monic $g \in \mathbb{Q}_p[t]$ sat.

1) $\deg(g) = \deg(f) = [K:\mathbb{Q}_p]$

& 2) $\|f - g\| < \varepsilon$

is irreducible.

— Moreover any such f, g have the "<u>root</u> <u>exchange</u> <u>property</u>".

Thus, we may find a <u>root</u> β of g s.t.

$$K = \mathbb{Q}_p(\alpha) = \mathbb{Q}_p(\beta).$$

Also, we may take $g \in \mathbb{Q}[t]$ since $\mathbb{Q} \subseteq \mathbb{Q}_p$ dense.
↖ <u>rational</u> coeffs.

Then $\beta \in \overline{\mathbb{Q}}$ is an alg. number,

$F = \mathbb{Q}(\beta)$ works.!

2. <u>Application</u>: $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$ is <u>alg.</u> <u>closed</u>.

$f \in \mathbb{C}_p[t]$ irr., monic. say $f(\alpha) = 0$ some $\alpha \in \overline{\mathbb{C}_p}$.

Find $\varepsilon > 0$ s.t. property (A) holds.

Since $\overline{\mathbb{Q}_p} \subseteq \mathbb{C}_p$ dense, there's a $g \in \overline{\mathbb{Q}_p}[t]$:

1) $\deg(g) = \deg(f)$ (<u>irreducible</u>, by Prop.)

& 2) $\|f - g\| < \varepsilon$.

From "<u>root</u> <u>exchange</u>" $\exists$ root β of g with

$$\mathbb{C}_p(\alpha) = \mathbb{C}_p(\beta)$$

However, $\beta \in \overline{\mathbb{Q}_p}$ so the RHS is $\mathbb{C}_p$. Ergo $\alpha \in \mathbb{C}_p$.

<u>Exc</u> $\overline{\mathbb{Q}_p}$ not complete.

$\xi_n =$ primitive $(p^{n!}-1)^{st}$ root of 1.

(so $\mathbb{Q}_p(\xi_n) = \mathbb{Q}_{p^{n!}}$ unram. of deg. $n!$,

<u>increasing</u>: $\mathbb{Q}_{p^{n!}} \subseteq \mathbb{Q}_{p^{(n+1)!}}$)

$$\sum_{n=0}^{\infty} \xi_n p^n$$: partial sums Cauchy, <u>not</u> convergent.

<u>Higher ramification:</u>

E/K finite Galois ext. of local fields.

$G = \mathrm{Gal}(E/K)$.

<u>Def.</u> $G_i = \{\tau \in G :$ τ acts <u>trivially</u> on $\mathcal{O}_E/\mathfrak{m}_E^{i+1}\}$

i.e., $v_E(\tau(x) - x) > i \quad \forall x \in \mathcal{O}_E$.

- All <u>normal</u> in G:

$G_i = \ker(G \to \mathrm{Aut}(\mathcal{O}_E/\mathfrak{m}_E^{i+1})) \triangleleft G$.

<u>Ex</u> $(i=0)$:

$G_0 = \ker(G \twoheadrightarrow \mathrm{Gal}(k_E/k_K))$

"<u>inertia</u>" $|G_0| = e(E/K)$.

<u>Filtration:</u> $G \supset G_0 \supset G_1 \supset G_2 \supset \cdots$

<u>Note:</u> $G_i = \{1\}$ for $i \gg 0$.

- Why? Stationary since $|G| < \infty$.

$\bigcap_{i=0}^{\infty} G_i = \{1\}$ since $\mathcal{O}_E \cong \varprojlim \mathcal{O}_E/\mathfrak{m}_E^i$.

Understand subquotients. ✓(1) $G/G_0 \cong \mathrm{Gal}(k_E/k_K) = \langle \varphi \rangle$ cyclic of order $f = f(E/K)$.

(2) Will relate G_i/G_{i+1} to $U_E^{(i)}/U_E^{(i+1)}$ for $i \geq 0$.

(3) Show $G_1 =$ <u>wild</u> inertia.

- Here, "<u>higher units</u>"

<u>Def.</u> $U_E^{(i)} = 1 + \mathfrak{m}_E^i$

$= \ker(\mathcal{O}_E^\times \to (\mathcal{O}_E/\mathfrak{m}_E^i)^\times)$.

Lemma. Suppose $\sigma \in G_0$, and $i \geq 0$. Then:

$$\sigma \in G_i \iff \frac{\sigma(\pi)}{\pi} \equiv 1 \pmod{\mathfrak{m}_E^i}$$

($\pi \in E$ any uniformizer).

Note: $|\sigma(\pi)| = |\pi|$ so $\frac{\sigma(\pi)}{\pi} \in \mathcal{O}_E^\times$.

PROOF. $\Rightarrow$: Assuming $v_E(\sigma(x) - x) > i \ \forall x \in \mathcal{O}_E$. $x = \pi$:

$$v_E\left(\frac{\sigma(\pi)}{\pi} - 1\right) = \underbrace{v_E(\sigma(\pi) - \pi)}_{>i} - \underbrace{v_E(\pi)}_{=1} > i-1.$$

$\Leftarrow$: Let $F = E^{G_0}$ be the max. unram. subext.

- Know: $\{1, \pi, \pi^2, \dots, \pi^{e-1}\}$ form an $\mathcal{O}_F$-basis $\mathcal{O}_E$.

(cf. proof $\mathcal{O}_E$ free/$\mathcal{O}_F$)

$$\begin{array}{c} E \\ \big| \text{ tot. ram.} \\ F \\ \big| \text{ ur.} \\ K \end{array}$$

Consider $x = a\pi^m$ with $a \in \mathcal{O}_F$.

- Then, since $\sigma(a) = a$,

$$\sigma(x) - x = a(\sigma(\pi)^m - \pi^m)$$

$$= a \underbrace{\left(\frac{\sigma(\pi)^m}{\pi^m} - 1\right)}_{\in\, \mathfrak{m}_E^i} \underbrace{\pi^m}_{\in\, \mathfrak{m}_E^m} \text{ lies in } \mathfrak{m}_E^{i+m} \subseteq \mathfrak{m}_E^{i+1} \text{ for } m > 0.$$

[if $m = 0$, $\sigma(x) - x = 0$].

$\square$

<u>Note</u>: $v_E\left(\frac{\sigma(u)}{u}-1\right)=v_E(\sigma(u)-u)>i$ for $\sigma\in G_i$.

$(u\in\mathcal{O}_E^\times)$

$v_E(u)=0$.

— i.e., $\frac{\sigma(u)}{u}\in U_E^{(i+1)}$.

$\Rightarrow$ the <u>coset</u> of $\frac{\sigma(\pi)}{\pi}$ in $U_E^{(i)}/U_E^{(i+1)}$ is <u>independent</u> of π.

Thus, the map

$$\begin{aligned}\psi: G_i &\longrightarrow U_E^{(i)}/U_E^{(i+1)}\\ \sigma &\longmapsto \frac{\sigma(\pi)}{\pi}\cdot U_E^{(i+1)}\end{aligned}$$

is a <u>homomorphism</u>

(indep. of π):

$$\frac{\sigma\tau(\pi)}{\pi}=\frac{\sigma(\tau(\pi))}{\tau(\pi)}\cdot\frac{\tau(\pi)}{\pi}\equiv\frac{\sigma(\pi)}{\pi}\cdot\frac{\tau(\pi)}{\pi}.$$

new uniformizer

By the lemma,

$\ker(\psi)=G_{i+1}$.

So,

$$\boxed{G_i/G_{i+1}\hookrightarrow U_E^{(i)}/U_E^{(i+1)}}.$$

is <u>injective</u> <u>homomorphism</u>.

<u>Will show</u>: $U_E^{(i)}/U_E^{(i+1)}\cong\begin{cases}k_E^\times & i=0.\\ k_E & i>0.\\ \text{(add. group)} & \end{cases}$