WEEK

8.

($p = \mathrm{char}(k)$)

– Consequently:
- G_0/G_1 is <u>cyclic</u> of prime–to–p order.
- G_i/G_{i+1} elementary abelian p-group. ($i>0$) $\simeq \mathbb{F}_p \oplus \cdots \oplus \mathbb{F}_p$.

Follows that G_1 is the Sylow p-subgroup of G_0.
(<u>normal</u> in G) "<u>wild</u> <u>inertia</u>".

notation switch: $E = K$.

<u>Theorem</u>. (1) $\mathcal{O}_K^\times / U_K^{(1)} \underset{\text{can.}}{\simeq} k^\times$

(2) $U_K^{(i)} / U_K^{(i+1)} \underset{\text{non-can.}}{\simeq} k \simeq \underbrace{\mathbb{F}_p \oplus \cdots \oplus \mathbb{F}_p}_{[k:\mathbb{F}_p] \text{ copies}}$. "elem. ab. p-gp."

with $i>0$.

PROOF. Recall

(1)
$$0 \to U_K^{(1)} \to \mathcal{O}_K^\times \longrightarrow k^\times \to 0$$
splits, canonically by HENSEL.

[∴ even: $\mathcal{O}_K^\times \underset{\text{can.}}{\simeq} k^\times \times U_K^{(1)}$.]

(2) Def. $\varphi : U_K^{(i)} \longrightarrow k$, $\varphi(u) = \varphi(1+\pi^i x) = \overline{x}$.

dep. π., surjective ✓.

<u>Homomorphism</u>:
- $\varphi(u) + \varphi(v) = \varphi(1+\pi^i x) + \varphi(1+\pi^i y) = \overline{x} + \overline{y}$.
- $\varphi(uv) = \varphi((1+\pi^i x)(1+\pi^i y)) = \varphi(1+\pi^i x + \pi^i y + \pi^{2i} xy) = \varphi(1+\pi^i(x+y+\pi^i xy)) = \overline{x+y+\pi^i xy} = \overline{x+y}$ ✓

& $\ker(\varphi) = U_K^{(i+1)}$ ✓

□

Local unit theorem: $K/\mathbb{Q}_p$ finite. Then,

$$U_K^{(1)} \simeq \mu_{p^\infty}(K) \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]} \quad \text{(non-can.)}$$

(finite) cyclic p-gp.

Note: Fails if $\mathrm{char}(K) = p$:

$$U_K^{(1)} \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots = \mathbb{Z}_p^{\mathbb{N}}.$$

Corollary: ($\mathrm{char}(K) = 0$)

$$\begin{aligned} \mathcal{O}_K^\times &\simeq \mu_{q-1}(K) \times U_K^{(1)} \\ &\simeq \mu_{q-1}(K) \times \mu_{p^\infty}(K) \times \mathbb{Z}_p^n \\ &\simeq \mu_{p'}(K) \times \mu_{p^\infty}(K) \times \mathbb{Z}_p^n \\ &\simeq \mu_\infty(K) \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}. \end{aligned}$$

and $K^\times \simeq \mathbb{Z} \times \mu_\infty(K) \times \mathbb{Z}_p^n$.

— in particular:

$$K^\times / K^{\times m} \simeq \mathbb{Z}/m\mathbb{Z} \times \mu_\infty / \mu_\infty^m \times (\mathbb{Z}_p / m\mathbb{Z}_p)^n$$

is finite, of order:

$$\boxed{|K^\times / K^{\times m}| = \frac{m}{\|m\|_K} \cdot |\mu_m(K)|}$$

— why? $\longrightarrow$

From $\simeq$, putting $w := |\mu_\infty(K)|$,

$$|K^\times / K^{\times m}| = m \cdot \mathrm{GCD}(m,w) \cdot p^{v_p(m) n}.$$

simplified: $\mu_\infty = \langle \gamma \rangle$. Then

$$(\gamma^r)^m = 1 \iff w \mid rm \iff \frac{w}{\mathrm{GCD}} \,\Big|\, r \frac{m}{\mathrm{GCD}} \iff \frac{w}{\mathrm{GCD}} \,\Big|\, r.$$

shows $|\mu_m(K)| = \mathrm{GCD}(m,w)$.

Moreover, $\|p\|_K = p^{-n}$ so $\|m\|_K^{-1} = p^{v_p(m) n}$. ✓

$\leftarrow$ ($n = ef$, $q = p^f$, $p \sim \pi^e$)

Ex. $|\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2}| = \frac{2}{1} \cdot 2 = 4$.

$p > 2$.

<u>three</u> quadratic extns.: pick $u \not\equiv x^2 \pmod p$,

$$\mathbb{Q}_p(\sqrt{u}) = \mathbb{Q}_{p^2}$$

$$\left.\begin{array}{l} \mathbb{Q}_p(\sqrt{p}) \\ \mathbb{Q}_p(\sqrt{up}) \end{array}\right\} \text{tot. \& tamely ram.}$$

– say $K = \mathbb{F}((t))$

$\star$) <u>Obs</u>: When $\mathrm{char}(K) = p$, $|K^\times / K^{\times p}| = \infty$.

– Otherwise also $\mathcal{O}_K^\times / \mathcal{O}_K^{\times p}$ is <u>finite</u>.

$$\mathcal{O}_K^\times \xrightarrow[\text{cont.}]{\text{image}} \mathcal{O}_K^\times$$
$$a \longmapsto a^p$$

$\mathcal{O}_K^\times$ compact $\Rightarrow$ $\mathcal{O}_K^{\times p}$ compact, <u>closed</u>.

By assumption $\mathcal{O}_K^{\times p}$ has finite index, so in fact must be <u>open</u> in $\mathcal{O}_K^\times$. Thus, $1 + t^N \in \mathcal{O}_K^{\times p}$ for all $N \gg 0$.

However, since

$\mathcal{O}_K = \mathbb{F}[[t]]$, $\mathcal{O}_K^p = \mathbb{F}[[t^p]]$.

<u>Contradiction</u> if $p \nmid N$.

<u>FACT</u>: Formula for $|K^\times / K^{\times m}|$ holds for $K = \mathbb{F}((t))$ w. convention that $\|m\|_K = 0$ when $p \mid m$.

(i.e., $m = 0$ in K)

– Analogous arguments show:

- $K^{\times m}$ <u>open</u> subgroup of <u>finite</u> index in $K^\times$, ~~provided $m \neq 0$ in K. (automatic if $K/\mathbb{Q}_p$)~~.

  [consider image of cont. m-power map
  $$(\cdot)^m : \mathcal{O}_K^\times \to \mathcal{O}_K^\times \text{]}$$

- If E/K finite, the <u>norm</u> group $N_{E/K}(E^\times)$ is <u>open</u> of <u>finite</u> <u>index</u> in $K^\times$, ~~provided $[E:K] \neq 0$ in K~~.

  [$K^{\times n} \subseteq N(E^\times)$]
  $n = [E:K]$

(for p-adic top.)

<u>Remark</u>: $K^{\times m}$ do <u>not</u> form a <u>neighborhood basis</u> at $1 \in K^\times$.

$$\left(K^{\times m} = \underbrace{m\mathbb{Z}} \times \mathcal{O}_K^{\times m} \text{ not in } \{0\} \times U_K^{(r)} \text{ for } m \gg 0 \right)$$

but $\{\mathcal{O}_K^{\times m}\}$ <u>does</u>, since $\mathcal{O}_K^\times / U_K^{(r)}$ finite ($m =$ index).

∞ idx in $K^\times$ though.

<u>Prop.</u> Let $K/\mathbb{Q}_p$ finite extn. Then every finite index $H \leqslant K^\times$ is automatically <u>open</u>, provided $\forall K$: index $\neq 0$ in K.

PF. Let $m := [K^\times : H]$ so that $K^{\times m} \leqslant H$.

Consider $u \in \mathcal{O}_K^\times$. <u>Claim</u>: If $u \in U_K^{(r)}$ for $r \gg 0$, "Newton's method" applies to $f(x) = x^m - u$ and $\alpha_0 = 1$.

<u>Check</u>: $|f(1)| = |1-u| < |f'(1)|^2 = |m|^2 \neq 0$.

ok if $u \equiv 1 \pmod{m_K^r}$ and $r \gg 0$.

NEWTON gives us an $\alpha \in \mathcal{O}_K$ s.t. $\alpha \equiv 1 \pmod{m_K}$

and $f(\alpha) = \alpha^m - u = 0$.

I.e., $U_K^{(r)} \subseteq \mathcal{O}_K^{\times m}$ for $r \gg 0$ dep. on m.

$\subseteq H$. $\square$

When $K = \mathbb{F}((t))$ there are finite index $H \leqslant K^\times$, — which are not open.

Construction: $U_K^{(1)} \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots$

— so $K^\times \simeq \mathbb{Z} \times \mu_{q-1}(K) \times \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots$

Consider projection $\varphi : K^\times \twoheadrightarrow \mathbb{F}_p \times \mathbb{F}_p \times \cdots$
(cont., open, gp. hom.)

$\cup|$ dense.

$\mathbb{F}_p \oplus \mathbb{F}_p \oplus \cdots$

no proper closed intermediate subgroup.

Pick any (proper) intermediate subgroup of finite index, V.
↳ cannot be closed.

Then $\varphi^{-1}(V) \leqslant K^\times$
finite index,
not open:

— otherwise

$V = \varphi(\varphi^{-1}(V))$ would be open.

eg. $V = \ker(\lambda)$

$$\prod \mathbb{F}_p \Big/ \bigoplus \mathbb{F}_p \xrightarrow{\lambda} \mathbb{F}_p$$

(any functional) on uncountable $\mathbb{F}_p$-vs.
(pick basis, take a coordinate)

__Exponential map:__ $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ convergent iff $\frac{|x|^n}{|n!|} \longrightarrow 0$.

$K/\mathbb{Q}_p$ finite.

($\mathrm{char}(K)=0$)

so $\frac{1}{n!} \in K$.

i.e., $v_K(x)n - v_K(n!) \to \infty$.

__Exc__ (de Polignac)

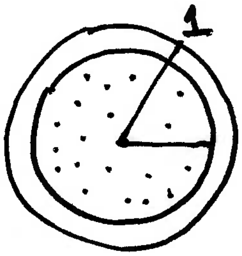
$$v_p(n!) = \sum_{r=1}^{\infty} \left[\frac{n}{p^r}\right] = \frac{n-s}{p-1}$$, $s =$ sum of p-adic digits of n.

— in particular $v_p(n!) < \frac{n}{p-1}$ for $n \geq 1$.

Therefore, noting $v_K(x) = e \cdot v_p(x)$, $e = e(K/\mathbb{Q}_p)$:

$$v_K(x)n - v_K(n!) > v_K(x)n - \frac{en}{p-1} = n\left(v_K(x) - \frac{e}{p-1}\right).$$

Shows $\exp(x)$ __converges__ for $v_K(x) > \frac{e}{p-1}$.



— i.e., in the __disk__: $|x| < q^{-e/(p-1)}$.

($\subseteq$ unit disk)

(normalized) abs. val.

(*) __Summary:__ $\exp(x)$ defined for $x \in \mathbb{C}_p$ with $|x|_p < p^{-1/(p-1)}$.

__Note:__ The normalized $|\cdot|_K$ is __not__ the one ext. $|\cdot|_p$ on $\mathbb{Q}_p$!

$|p|_K = q^{-e} = p^{-n}$, $n = ef = [K:\mathbb{Q}_p]$.

I.e., $|x|_K = |x|_p^n$ for all $x \in K$.

- Furthermore (by exc.): $v_p(n!) \leqslant \frac{n-1}{p-1}$ for $n \geqslant 1$.

So, for such n,

$$v_K\left(\frac{x^n}{n!}\right) = v_K(x)n - v_K(n!) \geqslant v_K(x)n - e\cdot\frac{n-1}{p-1} =$$

$$n\left(v_K(x) - \frac{e}{p-1}\right) + \frac{e}{p-1} > v_K(x)$$

[just check $n(\square) > \square$] as long as $\square > 0$ and $n > 1$.

$$\Longrightarrow \quad v_K\left(\exp(x) - (1+x)\right) > v_K(x)$$

$$\Longrightarrow \quad v_K\left(\exp(x) - 1\right) = \overset{\min}{\max}\{\ldots\} = v_K(x).$$

I.e., for $v_K(x) > \frac{e}{p-1}$, $\exp(x) \in 1 + m_K^{v_K(x)}$.

<u>Proposition</u>. Fix an integer $r > \frac{e}{p-1}$. Then

$$\exp : m_K^r \longrightarrow 1 + m_K^r = U_K^{(r)}$$

is a <u>continuous</u> <u>homomorphism</u>.
(vary r) - Moreover, $|\exp(x) - 1| = |x|$.

Logarithm: $\log(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ convergent iff $\frac{|x-1|^n}{|n|} \to 0$.

($K/\mathbb{Q}_p$ finite)

$\frac{1}{n} \in K$.

i.e., $v_K(x-1)n - v_K(n) \to \infty$.

– Now, as $v_K(n) = e \cdot v_p(n)$,

$$v_K(x-1)n - v_K(n) \geqslant v_K(x-1)n - \frac{e}{\log(p)} \cdot \log(n) \quad (*)$$

– why is $v_p(n) \leqslant \frac{\log(n)}{\log(p)}$?: [Factor $n = p^{v_p(n)} m \geqslant p^{v_p(n)}$ take log.

$(*) \to \infty$ as $n \to \infty$,

provided $v_K(x-1) > 0$, so $\log(x)$ is defined in the disk $|x-1| < 1$. – even: ($x \in \mathbb{C}_p$)

(i.e. for $x \in 1 + \mathfrak{m}_K = U_K^{(1)}$)

– Furthermore, for $n \geqslant 1$,

$$\begin{aligned} v_K\left(\tfrac{1}{n}(x-1)^n\right) &= n v_K(x-1) - v_K(n) \\ &= n v_K(x-1) \underbrace{- v_K(n!)}_{\geqslant -e\frac{n-1}{p-1}} + \underbrace{v_K((n-1)!)}_{\geqslant 0} \\ &\geqslant n v_K(x-1) - e\frac{n-1}{p-1} \end{aligned}$$

$$= n\left(v_K(x-1) - \frac{e}{p-1}\right) + \frac{e}{p-1} > v_K(x-1)$$

[of the form $n(\odot) > \odot$]

- provided $n \geq 2$ and $v_K(x-1) - \frac{e}{p-1} > 0$.

$\sim$ Under these assumptions,

$$v_K(\log(x) - (x-1)) > v_K(x-1) \implies$$

$$v_K(\log(x)) = \min\{\dots\} = v_K(x-1).$$

I.e., for $v_K(x-1) > \frac{e}{p-1}$, $\log(x) \in \mathfrak{m}_K^{v_K(x-1)}$.

<u>Prop.</u> Fix an integer $r > \frac{e}{p-1}$. Then

$$\log : U_K^{(r)} = 1 + \mathfrak{m}_K^r \longrightarrow \mathfrak{m}_K^r$$

is a <u>continuous</u> <u>homomorphism</u>. Moreover, $|\log(x)| = |x-1|$.

<u>Combined</u>: $\forall r > \frac{e}{p-1}$,

$$\mathfrak{m}_K^r \underset{\log}{\overset{\exp}{\rightleftarrows}} U_K^{(r)}$$

mutually inverse $\simeq$ of <u>top.</u> <u>groups</u>.

∘ p-<u>powers</u>: $\forall\, u = 1+\pi^r x \in U_K^{(r)}$,

$$u^p = \sum_{s=0}^{p} \binom{p}{s} \pi^{rs} x^s = 1 + \sum_{s=1}^{p} (\cdots)$$

– For $s>0$, $\pi^{rs} \in \mathfrak{m}_K^r$ and $\binom{p}{s} \in (p) = \mathfrak{m}_K^e$ <u>provided</u> $0<s<p$.

The p^{th} term $(s=p)$:

$\pi^{rp} x^p$ also lies in $\mathfrak{m}_K^{r+e}$ since $rp \geq r+e$ (as $r > \frac{e}{p-1}$).

Conclude $u^p \in U_K^{(r+e)}$.

<u>Corollary</u>: For $r > \frac{e}{p-1}$, $u \mapsto u^p$ gives an <u>isomorphism</u> $U_K^{(r)} \xrightarrow{\sim} U_K^{(r+e)}$ (top. gps.)

PROOF.

$$\begin{tikzcd}
U_K^{(r)} \arrow[r, "u \mapsto u^p"] \arrow[d, "\log\ \exp"', "\simeq"] & U_K^{(r+e)} \\
\mathfrak{m}_K^r \arrow[r, "x \mapsto px", "\simeq"'] & \mathfrak{m}_K^{r+e} \arrow[u, "\simeq", "\exp"']
\end{tikzcd}$$

$\square$

<u>Corollary</u> $\log : U_K^{(1)} \to K$ has <u>kernel</u> $\mu_{p^\infty}(K)$.

[note: $\mu_{p^\infty}(K) \leqslant U_K^{(1)}$ since $K^\times = \mathbb{Z} \times \underbrace{\mu_{q-1}(K)}_{\mu_{p'}(K)} \times U_K^{(1)}$.]

— more precisely $\mu_{p^\infty}(K) = \mu_\infty(K) \cap U_K^{(1)}$.

PROOF. First suppose $u \in U_K^{(1)}$ and $u^N = 1$, some $N \geq 1$. then,

$N \cdot \log(u) = 0$, so $\log(u) = 0$ ✓.

— Second, suppose $u \in U_K^{(1)}$ and $\log(u) = 0$.

$m := [U_K^{(1)} : U_K^{(r)}]$ for some choice of $r > \frac{e}{p-1}$.

(p — power).

Since $u^m \in U_K^{(r)}$ and $\log(u^m) = m \log(u) = 0$,

bijectivity of log for such r tells us $u^m = 1$. □

— Argument also shows: $\mu_{p^\infty}(K) = \mu_{p^r}(K)$ with q^r

$r > \frac{e}{p-1}$ (minimal).

$K/\mathbb{Q}_p$.

<u>Note:</u> $U_K^{(r)}$ is a (multiplicative) $\mathbb{Z}_p$-module, via

$$u^s = (1+\pi^r x)^s = \sum_{t=0}^{\infty} \binom{s}{t} \pi^{rs} x^s$$

find $s_n \to s$ with $s_n \in \mathbb{N}$

for $s \in \mathbb{Z}_p$. Here $\binom{s}{t} = \frac{1}{t!} s(s-1)\cdots(s-t+1) \in \mathbb{Z}_p$.

$\Rightarrow$ convergent.

<u>Local Unit Thm:</u> (char$(K)=0$)

$$U_K^{(1)} \simeq \mu_{p^\infty}(K) \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}.$$

finite cyclic p-gp.

PROOF. Found

$$\log : U_K^{(1)}/\mu_{p^\infty}(K) \hookrightarrow \mathfrak{m}_K \simeq \mathcal{O}_K \simeq \mathbb{Z}_p^{[K:\mathbb{Q}_p]}.$$

($\mathbb{Z}_p$-linear). Since $\mathbb{Z}_p$ is PID, $U_K^{(1)}/\mu_{p^\infty}(K)$ is free of rank $\leq [K:\mathbb{Q}_p]$ over $\mathbb{Z}_p$.

On the other hand,

$$U_K^{(r)} \hookrightarrow U_K^{(1)}/\mu_{p^\infty}(K)$$

for $r > \frac{e}{p-1}$ [in this range $U_K^{(r)} \simeq \mathfrak{m}_K^r = \pi^r \mathcal{O}_K \simeq \mathcal{O}_K$ free of rank $[K:\mathbb{Q}_p]$.]

In conjunction,

$U_K^{(1)}/\mu_{p^\infty}(K)$ is free$/\mathbb{Z}_p$ of rank $[K:\mathbb{Q}_p]$.

From <u>structure thm.</u> for f.g. $\mathbb{Z}_p$-mods.,

$$U_K^{(1)} \simeq \underbrace{(\text{finite } p\text{-gp.})}_{\mu_{p^\infty}(K)} \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]} \qquad \square$$

— Found $U_K^{(r)} \simeq m_K^r \simeq \mathcal{O}_K \simeq \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$ for $r > \frac{e}{p-1}$.

basis?

<u>Recall:</u> $U_K^{(r)} \xrightarrow[u \mapsto u^p]{\sim} U_K^{(r+e)}$,

with

$$\dim_{\mathbb{F}_p} U_K^{(r)} / U_K^{(r+e)} = n = [K:\mathbb{Q}_p].$$

in particular $U_K^{(r)}/U_K^{(r+e)}$ is an $\mathbb{F}_p$-vector space (mult.) of <u>cardinality</u> $q^e = p^n$ $(n = ef = [K:\mathbb{Q}_p])$

Choose $\mathbb{F}_p$-basis $\{\bar{u}_1, \dots, \bar{u}_n\}$, where all $u_i \in U_K^{(r)}$.

<u>Thm</u>. $\{u_1, \dots, u_n\}$ is a $\mathbb{Z}_p$-basis for $U_K^{(r)}$.

(i.e., $u = u_1^{s_1} \cdots u_n^{s_n}$ uniquely, $s_i \in \mathbb{Z}_p$).

PROOF. Introduce notation $M := U_K^{(r)}$ (additive), $N := \mathbb{Z}_p^n$

$$f : N \longrightarrow M$$

$$(s_1, \dots, s_n) \longmapsto u_1^{s_1} \cdots u_n^{s_n} = \sum_{i=1}^n s_i u_i$$

isomorphism?

<u>Note</u>: $M/pM = U_K^{(r)}/U_K^{(r+e)}$ since $r > \frac{e}{p-1}$.

Claim: $N/p^iN \longrightarrow M/p^iM$ <u>isomorphism</u> $\forall i > 0$.

($i = 1$: $\mathbb{F}_p^n \xrightarrow{\sim} M/pM$ since $\{\bar{u}_i\}$ basis)

Induction: Apply "snake lemma" to the diagram

$$\begin{tikzcd} 0 \arrow[r] & N/pN \arrow[r, "p^i"] \arrow[d, "\wr"] & N/p^{i+1}N \arrow[r] \arrow[d] & N/p^iN \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & M/pM \arrow[r, "p^i"] & M/p^{i+1}M \arrow[r] & M/p^iM \arrow[r] & 0 \end{tikzcd}$$

(know M free$/\mathbb{Z}_p$).

Finally pass to completions,

$$f: M = \varprojlim M/p^iM \xrightarrow{\sim} N = \varprojlim N/p^iN \quad \square$$

Ex $K = \mathbb{Q}_p$ with $p > 2$. (Here $\frac{e}{p-1} = \frac{1}{p-1} < 1$ so $r = 1$ works)

$\mathbb{F}_p$-vs., $U^{(1)}_{\mathbb{Q}_p} / U^{(2)}_{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{F}_p$. — no μ_{p^∞} in $\mathbb{Q}_p$.

$[1+px] \longmapsto \bar{x}$

may take $u = 1+p$, $\mathbb{Z}_p$-basis for $U^{(1)}_{\mathbb{Q}_p}$.

Thus,

$$\mathbb{Q}_p^\times = p^{\mathbb{Z}} \times \mu_{p-1}(\mathbb{Q}_p) \times (1+p)^{\mathbb{Z}_p}.$$

[When $p=2$, $\frac{e}{p-1} = 1$ so need $r=2$. As above $U^{(2)}_{\mathbb{Q}_2} / U^{(3)}_{\mathbb{Q}_2} \simeq \mathbb{F}_2$. Get $\mathbb{Z}_2$-basis $u = 1+2^2 = 5$ for $U^{(2)}_{\mathbb{Q}_2}$].

$[1+2^2x] \mapsto \bar{x}$

Leopoldt Conjecture: $F/\mathbb{Q}$ number field, p = prime

$\forall \mathfrak{p} | p$ consider $U_{F_\mathfrak{p}}^{(1)}$.

- embed global units:

$$\phi: \mathcal{O}_F^\times \xrightarrow{\text{diag}} \prod_{\mathfrak{p}|p} \mathcal{O}_{F_\mathfrak{p}}^\times .$$

Let

$$G = \phi^{-1}\Big(\prod_{\mathfrak{p}|p} U_{F_\mathfrak{p}}^{(1)}\Big).$$

finite index $\leq \mathcal{O}_F^\times$, so f.g. of rank $r_1 + r_2 - 1$.

closure:

$$\overline{\phi(G)} \leq \prod_{\mathfrak{p}|p} U_{F_\mathfrak{p}}^{(1)}$$

$\mathbb{Z}_p$ - submodule.

Conj (Leopoldt): $\operatorname{rank}_{\mathbb{Z}_p} \overline{\phi(G)} = r_1 + r_2 - 1$.

(ok for abelian ext. $F/\mathbb{Q}$)