WEEK 9.

Local Class Field Theory.

K = local field (non-arch.)

char$(K) = 0$: $K/\mathbb{Q}_p$ finite.

char$(K) > 0$: $K/\mathbb{F}_p((t))$

p = residual characteristic.

$\overline{K} = K^{sep}$

$\mathrm{Gal}(\overline{K}/K) = \varprojlim \mathrm{Gal}(E/K)$ profinite.

[with E/K fin. Galois. $E \subseteq \overline{K}$.]

* GOAL

Understand:

$$K^{ab} = \bigcup_{\substack{E/K \\ \text{abelian}}} E$$

"max. abelian extn. of K".

(i.e., fin. Galois w. $\mathrm{Gal}(E/K)$ abelian)

$$\mathrm{Gal}(K^{ab}/K) = \varprojlim_{E/K \text{ abelian}} \mathrm{Gal}(E/K) = \mathrm{Gal}(\overline{K}/K)^{ab} = \text{largest abelian Hausdorff quotient.}$$

(mod out by closure of commutators).

<u>Note</u>: $K^{ur} \subseteq K^{ab}$.

Indeed,

$$\mathrm{Gal}(K^{ur}/K) \simeq \varprojlim \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \simeq \widehat{\mathbb{Z}}$$

is pro-cyclic (top. gen. by Frobenius).

Main Thm ("Local Reciprocity").

$\exists!$ cont. homomorphism — known as the Artin map,

$$\phi = \phi_K : K^\times \longrightarrow \mathrm{Gal}(K^{ab}/K),$$

satisfying: (a) $\forall$ fin. unramified E/K, $\pi \in K$ any uniformizer.

$$\phi(\pi)\big|_E = \mathrm{Frob}_{E/K}.$$

(b) $\forall$ fin. abelian E/K,

$\phi(N_{E/K}(E^\times)) \subseteq \mathrm{Gal}(K^{ab}/E)$, and ϕ induces

$$\phi_{E/K} : K^\times / N_{E/K}(E^\times) \xrightarrow{\sim} \mathrm{Gal}(E/K).$$

Remark: From (a) — since π is arbitrary — get

$$\phi(\mathcal{O}_K^\times) \leq \mathrm{Gal}(K^{ab}/K^{ur}) = I_K^{ab}.$$

Thm ("Local Existence").

Every open $N \leq K^\times$ of finite index is of the form $N = N_{E/K}(E^\times)$ for some fin. abelian E/K,

"norm groups" and vice versa.

proved thrs.

- Together they yield:

<u>Classification Thm</u>. There's a <u>bijection</u>,

$$\left\{\begin{matrix}\text{fin. abelian}\\ \text{extns. } E/K\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{fin. index}\\ \text{open } N \leqslant K^\times\end{matrix}\right\}.$$

$$E \longmapsto N_{E/K}(E^\times)$$

- Why <u>injective</u>? From (b) know $\phi\big(N(E^\times)\big) \overset{\text{dense}}{=} \mathrm{Gal}(K^{ab}/E)$.

- I.e., $E = (K^{ab})^{\phi(N)}$, $N = N(E^\times)$.

$N(E) \overset{\text{df.}}{=} N_{E/K}(E^\times)$ (recover E from N by taking <u>invariants</u> in K^{ab}) ✓

- <u>Further properties</u>: (0) $[E:K] = [K^\times : N]$.

(1) $E_1 \subset E_2 \iff N(E_2) \subset N(E_1)$

(inclusion-reversing)

(2) $N(E_1E_2) = N(E_1) \cap N(E_2)$

(3) $N(E_1 \cap E_2) = N(E_1) \cdot N(E_2)$

(4) K_n = the <u>unr.</u> deg n extn.

$$K_n \longleftrightarrow N(K_n) \underset{\substack{\uparrow\\ \text{why?:}}}{=} \pi^{n\mathbb{Z}} \times \mathcal{O}_K^\times$$

— By (b):

$$\phi_{K_n/K} : K^\times / N(K_n) \xrightarrow{\sim} \mathrm{Gal}(K_n/K) = \langle \mathrm{Frob}_{K_n/K} \rangle .$$

takes

$$[\pi] \overset{(a)}{\longmapsto} \mathrm{Frob}_{K_n/K} .$$

$$[u\pi^v] \longmapsto \mathrm{Frob}_{K_n/K}^v .$$

$(u \in \mathcal{O}_K^\times)$

<u>Shows</u>.

$$N(K_n) = \{ u\pi^v \mid \underset{(\text{i.e., } n \mid v)}{\mathrm{Frob}_{K_n/K}^v = 1} \} = \pi^{n\mathbb{Z}} \times \mathcal{O}_K^\times . \checkmark$$

<u>Corollary</u>: $\widehat{K^\times} \xrightarrow{\sim} \mathrm{Gal}(K^{ab}/K)$.

($K/\mathbb{Q}_p$ finite)

↳ showed all fin. idx. $N \leq K^\times$ are open.

profinite completion $(= \varprojlim_N K^\times/N)$.

— in this case: ∞ Galois Theory,

$$\left\{ \begin{matrix} \underline{\text{all}} \text{ abelian} \\ \text{extns. } E/K \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{closed subgroups} \\ N \leq \widehat{K^\times} \end{matrix} \right\} .$$

EX.

<u>Kummer Extensions</u>: Fix $n>1$, assume X^n-1 splits$/K$.

— suppose $n \neq 0$ in K. (so X^n-1 sep., and $|\mu_n(\overline{K})| = n$).

i.e., $\mu_n(\overline{K}) \leq K^\times$

Let $a \in K^\times$ and <u>pick</u> nth root $\sqrt[n]{a} \in \overline{K}$.

$E := K(\sqrt[n]{a})$ indep. of <u>choice</u> since $\mu_n \leq K^\times$.

(= splitting field of $X^n - a$, sep.).

— so GALOIS.

$$\psi : \mathrm{Gal}(E/K) \longrightarrow \mu_n$$

$$\tau \longmapsto \frac{\tau(\sqrt[n]{a})}{\sqrt[n]{a}} \quad \text{(well-def.)}.$$

injective homomorphism,

so E/K cyclic,

$\mathrm{Gal}(E/K)$ of "<u>exponent</u>" n, i.e. $\tau^n = 1 \ \forall \tau \in \mathrm{Gal}(E/K)$.

— more generally $E = K(\sqrt[n]{a_1}, \dots, \sqrt[n]{a_N})$ has

$$\mathrm{Gal}(E/K) \hookrightarrow \mu_n \times \cdots \times \mu_n = \mu_n^N.$$

~~cyclic~~ of exp. n.

abelian

$\forall$ $\Delta \leqslant K^\times / K^{\times n}$ (identified with $\tilde{\Delta} \leqslant K^\times$ cont. $K^{\times n}$, $\Delta = \tilde{\Delta}/K^{\times n}$). eg. $\tilde{\Delta} = \langle a \rangle K^{\times n}$.

– Consider:

$E = K(\sqrt[n]{\Delta})$ adjoin a <u>choice</u> of $\sqrt[n]{a}$ for <u>each</u> $a \in \tilde{\Delta}$.

Get <u>pairing</u>:

$$\mathrm{Gal}(E/K) \times \Delta \longrightarrow \mu_n$$
$$(\tau, a) \longmapsto \frac{\tau(\sqrt[n]{a})}{\sqrt[n]{a}}$$

non-degenerate (check), so

$$\mathrm{Gal}(E/K) \cong \mathrm{Hom}(\Delta, \bar{K}^\times) = \Delta^\vee.$$

[in particular $[E:K] = |\Delta|$, even if ∞]

doesn't happen if K <u>local</u> and $n \not\equiv 0$ in K.

<u>Kummer Theory</u>:

$$\left\{ \begin{array}{l} \text{subgroups} \\ \Delta \leqslant K^\times / K^{\times n} \end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{l} \text{abelian extns.} \\ E/K \text{ of exponent } n \end{array} \right\}$$
$$\Delta \longmapsto K(\sqrt[n]{\Delta})$$

~~index~~ card. ~ degree, inclusion – <u>preserving</u>.

(inverse: $E \mapsto \tilde{\Delta} = E^{\times n} \cap K^\times$)

- **Relation to LCFT:**
(assume $\mu_n \leq K^\times$ etc.)

E/K abelian of exponent n.

Kummer $\swarrow\!\!\nearrow$ increasing

decreasing $\nwarrow\!\!\searrow$ Class Field Theory.

$\tilde{\Delta} \leq K^\times$ containing $K^{\times n}$.

$N(E) \leq K^\times$ open, fin. idx.

It's not true that $N(E) = \tilde{\Delta}$! (~~although same index~~)

Thm. They're dual rel. to the "Hilbert symbol"; I.e. the perfect pairing

$$(,) : K^\times / K^{\times n} \times K^\times / K^{\times n} \longrightarrow \mu_n .$$

$$(\alpha, \beta) = \frac{\phi_K(\alpha)(\sqrt[n]{\beta})}{\sqrt[n]{\beta}}$$

– The <u>max</u> abelian extn. E/K (of exponent n): $\quad E_{max} = K(\sqrt[n]{K^\times})$.

$$\tilde{\Delta} = K^\times \;\overset{\text{Kummer}}{\longleftrightarrow}\; \text{(max abelian extn.)} \;\overset{\text{LCFT}}{\longleftrightarrow}\; \text{smallest } N$$

<u>smallest</u> $N \leqslant K^\times$ (open, fin. idx.) s.t. $K^\times/N$ killed by n, i.e. $N \supseteq K^{\times n}$.

$$N = K^{\times n}$$

– In other words,

$$N\left(K(\sqrt[n]{K^\times})\right) = K^{\times n}.$$

Thus the <u>Artin</u> <u>map</u> ϕ_K induces

$$\phi_{E/K}: K^\times/K^{\times n} \xrightarrow{\sim} \mathrm{Gal}(E_{max}/K)$$

Combined w. Kummer <u>pairing</u> above,

$$\begin{array}{ccccc} K^\times/K^{\times n} \otimes K^\times/K^{\times n} & \xrightarrow{\sim} & \mathrm{Gal}(\overbrace{E_{max}/K}^{\Delta_{max}}) \otimes K^\times/K^{\times n} & \longrightarrow & \mu_n \\ \alpha \otimes \beta & \longmapsto & \phi_K(\alpha) \otimes \beta & \longmapsto & \dfrac{\phi_K(\alpha)(\sqrt[n]{\beta})}{\sqrt[n]{\beta}}. \end{array}$$

$(\alpha, \beta \in K^\times)$ mod $K^{\times n}$

– <u>non</u> – <u>degenerate</u> bilinear pairing "HILBERT SYMBOL"

$$(\,,\,): K^\times/K^{\times n} \times K^\times/K^{\times n} \longrightarrow \mu_n$$

FACTS: (1) $(-\alpha,\alpha)=1$

(2) $(1-\alpha,\alpha)=1 \quad \forall \alpha \neq 1.$

(3) $(\alpha,\beta)=(\beta,\alpha)^{-1}$ symplectic.

Thm. V.II

Prop: $N\left(K(\sqrt[n]{\Delta})\right)=\Delta^{\perp}=\{\alpha\in K^{\times}: (\alpha,\beta)=1 \ \forall \beta\in\Delta\}.$

$\forall \Delta.$

PROOF. Let $\alpha\in K^{\times}$. Then,

$$\alpha\in N(K\sqrt[n]{\Delta}) \iff \phi_K(\alpha)=1 \text{ on } K(\sqrt[n]{\Delta})$$

$$\iff \phi_K(\alpha)(\sqrt[n]{\beta})=\sqrt[n]{\beta}, \ \forall \beta\in\tilde{\Delta}$$

$$\iff (\alpha,\beta)=1, \ \forall\beta\in\tilde{\Delta}. \quad \square$$

Lubin–Tate Theory:

Recall: $K^{ur} = \bigcup_{m>0} K_m$ ← unr. deg m.

– Construct totally ramified abelian extensions: $K_{\pi,n}$
(dep. choice of $\pi \in K$)

$$\begin{array}{c} K_{\pi,n} \\ | \\ K \end{array}$$

+ cont. homomorphism

$$\psi_\pi : K^\times \longrightarrow \mathrm{Gal}(K_\pi K^{ur}/K)$$

$K_\pi := \bigcup_{n>0} K_{\pi,n}$ (field)

sat. the properties below.

(0) $[K_{\pi,n} : K] = q^{n-1}(q-1)$

(1) $\psi_\pi(\pi)\big|_{K^{ur}} = \mathrm{Frob}_K$.

(2) $\underset{\pi^{m\mathbb{Z}} \times U_K^{(n)}}{\psi_\pi(\underset{\cap}{a})} = 1$ on $K_{\pi,n} K_m$. (all $m, n > 0$)

(3) $\pi \in N(K_{\pi,n})$ $\forall n > 0$.

(4) $K_\pi K^{ur}$ and ψ_π are independent of π.

(eventually: $K_\pi K^{ur} = K^{ab}$ and $\psi_\pi = \phi_K$)

Ex. ($K = \mathbb{Q}_p$, $\pi = p$) Here $\underset{K_{\pi,n}}{\mathbb{Q}_{p,n}} = \mathbb{Q}_p(\zeta_{p^n})$.

note $p = N(1 - \zeta_{p^n})$ $\forall n > 0$.

Lemma Suppose $\phi : K^\times \longrightarrow \mathrm{Gal}(K^{ab}/K)$ cont. hom.
satisfying local reciprocity.   exist?

Then: $K^{ab} = K_\pi K^{ur}$ and $\phi = \psi_\pi$.

- for $K = \mathbb{Q}_p$ this is "local Kronecker-Weber":
$$\mathbb{Q}_p^{ab} = \mathbb{Q}_p(\mu_\infty).$$

$\Rightarrow$ Artin maps are unique.

PROOF (Lemma): Introduce $\tilde{K} := K_\pi K^{ur}$ and $\tilde{\psi} := \psi_\pi$
(indep. of π by (4)).

1st obs.: $\forall a \in K^\times$,
$$\phi(a)|_{\tilde{K}} = \psi_\pi(a) = \tilde{\psi}(a). \qquad (*)$$

Why? Check this for $a = \pi$.

Know $\pi \in N(K_{\pi,n})$ $\forall n$ by (3). I.e., $\phi(\pi) = 1$ on $K_{\pi,n}$ all n

I.e., $\phi(\pi)|_{K_\pi} = 1$.

Also, by loc. reciprocity, $\phi(\pi)|_{K^{ur}} = \mathrm{Frob}_K$.

— same holds for ψ_π by (1).

Suffices to check $\psi_\pi(\pi) = 1$ on K_π.

This follows from the $m=1$ case of (2).

The previous argument works for *any* uniformizer π.
(Set of such generate $K^\times$: $a = u\pi^v = (u\pi)\pi^{v-1} = \pi' \pi^{v-1}$)
— this verifies (*).

NOTATION: $K_{n,m} = K_{\pi,n} K_m$
$(m,n>0)$
$U_{n,m} = \pi^{m\mathbb{Z}} \times U_K^{(n)}$

Claim: $N(K_{n,m}) = U_{n,m}$.

$\supseteq$: By (2): $\underset{\underset{U_{n,m}}{\cap}}{\psi_\pi(a)} = 1$ on $\underset{\underset{\psi}{U_{n,m}}}{K_{n,m}}$.

By (*) this is equivalent to $\phi(a) = 1$ on $K_{n,m}$,
which means $a \in N(K_{n,m})$ by loc. reciprocity.

Both sides same index in $K^\times$:

$$[K^\times : U_{n,m}] = [\pi^{\mathbb{Z}} : \pi^{m\mathbb{Z}}] \cdot [\mathcal{O}_K^\times : U_K^{(n)}]$$
$$= m \cdot (q-1) \cdot q^{n-1}$$
$$\overset{(0)}{=} [K_m : K] \cdot [K_{\pi,n} : K]$$
$$= [K_{n,m} : K]$$
$$= [K^\times : N(K_{n,m})]$$

$\leftarrow$ note:
$K_m \cap K_{\pi,n} = K$
unr & tot. ram.

E/K <u>any</u> abelian extn.

Know $N(E) \leq K^\times$ is <u>open</u> so $N(E) \supseteq U_{n,m}$ for <u>some</u> $m,n > 0$.

(eg. $\pi^m \in N(E)$, $m =$ index)

<u>Claim</u>: $E \subseteq K_{n,m}$.

Why? For $a \in K^\times$,

- $\phi(a)|_E = 1 \iff a \in N(E)$
- $\phi(a)|_{K_{n,m}} = 1 \iff a \in N(K_{n,m}) \xrightarrow{\text{claim \#1}} U_{n,m}$.

Since $U_{n,m} \subseteq N(E)$, $\quad \sigma|_{K_{n,m}} = 1 \Rightarrow \sigma|_E = 1$

for all $\sigma \in \mathrm{Gal}(EK_{n,m}/K)$

(surjective at fin. level: $\sigma = \phi(a)$)

i.e.,

$$\mathrm{Gal}(EK_{n,m}/K_{n,m}) \leq \mathrm{Gal}(EK_{n,m}/E)$$

Fixed fields (Galois Theory):

$$K_{n,m} \supseteq E. \checkmark$$

<u>CONCLUDE</u>: $K^{ab} = \tilde{K} = K_\pi K^{ur}$

and therefore also $\phi = \psi_\pi$ by $(*)$. $\square$

<u>Local Existence</u>: $N \leq K^\times$ open, fin. idx. $N \overset{?}{=} N(E)$.

$\exists m,n$:

$N \supseteq U_{n,m} = N(K_{n,m})$. , $\phi : K^\times \to \mathrm{Gal}(K_{n,m}/K)$.
Artin map.

Let $E := K_{n,m}^{\phi(N)}$. , finite abelian E/K

By CFT, for $a \in K^\times$,

- $a \in N(E) \iff \phi(a)|_E = 1$.

Moreover

- $a \in N \iff \phi(a)|_E = 1$.

obvious from def. of E.
(N – fixed subfield $\subseteq K_{n,m}$).

suppose
$\phi(a) \in \mathrm{Gal}(K_{n,m}/E) = \phi(N)$.
then
$a \in N \cdot \ker(\phi) = N \cdot U_{n,m} = N$.

Thus $N = N(E)$ ✓.

<u>Outline</u>: "Formal group law" $F \in R[\![X,Y]\!]$ (eventually $R = \mathcal{O}_K$).
(one-parameter)

(1) $F(X,Y) = X + Y + \cdots$

(2) $F(X, F(Y,Z)) = F(F(X,Y), Z)$

(3) $F(X, I(X)) = 0$ for unique $I(X) = X + \cdots$

<u>Ex</u> $F_a(X,Y) = X + Y$ and $F_m(X,Y) = (1+X)(1+Y) - 1$.

– Actual groups: $F \in \mathcal{O}_K[\![X,Y]\!]$, on $\mathfrak{m}_K$:
(K = local, non-arch.)

$$x \underset{F}{+} y := F(x,y) \text{ (convergent)}$$

(e.g. for F_m get $U_K^{(1)}$.)

<u>Homomorphisms</u>: $\phi(T) = aT + \cdots$ such that
$\phi: F \to G$ $\quad \phi(F(X,Y)) = G(\phi(X), \phi(Y))$

End(F) has <u>ring</u> structure ($1 = T$, composition, +)

<u>Def</u>. Fix $\pi \in K$. Then $\mathcal{F}_\pi$ is the collection of all
$f \in \mathcal{O}_K[\![X]\!]$ s.t. (a) $f(X) = \pi X + \cdots$
(b) $f(X) \equiv X^q \pmod{\pi}$

<u>Ex</u> $\pi X + X^q$

$K = \mathbb{Q}_p$: $(1+X)^p - 1 = pX + \binom{p}{2}X^2 + \cdots + X^p$

Thm. $\forall f \in \mathcal{F}_\pi$ there's a unique formal group law $F_f \in \mathcal{O}_K[\![X,Y]\!]$ s.t. $\mathrm{End}(F_f)$ contains f.

and any two $f, g \in \mathcal{F}_\pi$ have $F_f \simeq F_g$.

(in the $\mathbb{Q}_p$-ex. above $F_f = F_m$)

Moreover, F_f is a formal $\mathcal{O}_K$-module:

$$\begin{aligned} \mathcal{O}_K &\longrightarrow \mathrm{End}(F_f) \\ a &\longmapsto [a] = aT + \cdots \end{aligned}$$

(injective) $\qquad$ commuting with f. $\qquad$ EX $[\pi] = f$.

Def. Consider $\mathfrak{m}_{\overline{K}}$ with addition $x + y := F_f(x,y)$.

and $\mathcal{O}_K$-mod. structure: $a * x := [a](x)$.

$\forall n \geq 1$,

$$\Lambda_n = \{ x \in \mathfrak{m}_{\overline{K}} \mid \pi^n * x = 0 \}$$

π^n-torsion in $(\mathfrak{m}_{\overline{K}}, +_{F_f})$.

(dep. on f) $\qquad$ all roots of $f^{\circ n}$ in $\overline{K}$.

(automatically in $\mathfrak{m}_{\overline{K}}$) - Newton polygons.

EX ($K = \mathbb{Q}_p$, $f(x) = (1+x)^p - 1$, $F_f = \widehat{\mathbb{G}}_m$.

Here

$$\Lambda_n = \{ x \in \overline{\mathbb{Q}}_p \mid (1+x)^{p^n} - 1 = 0 \} \simeq \mu_{p^n}(\overline{\mathbb{Q}}_p)$$

Thm. (i) $\Lambda_n \simeq \mathcal{O}_K/(\pi^n)$
(non-can.)

(ii) The map $\mathcal{O}_K \to \mathrm{End}(\Lambda_n)$ induces can. iso.

$$\mathcal{O}_K/(\pi^n) \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_K}(\Lambda_n)$$

(iii) $K_{\pi,n} := K(\Lambda_n)$ satisfies:

(0) Tot. ram. of deg $[K_{\pi,n}:K] = (q-1)q^{n-1}$

(1) $(\mathcal{O}_K/(\pi^n))^\times \xrightarrow{\sim} \mathrm{Aut}(\Lambda_n) \xleftarrow{\sim} \mathrm{Gal}(K_{\pi,n}/K)$.

[in particular $K_{\pi,n}/K$ abelian]

(2) $\pi \in N(K_{\pi,n})$.

Cor. $\mathrm{Gal}(K_\pi/K) \xrightarrow{\sim} \mathcal{O}_K^\times$.

($K_\pi := \bigcup_{n>0} K_{\pi,n}$)

"**Artin map**": $\psi_\pi : K^\times \longrightarrow \mathrm{Gal}(K_\pi K^{ur}/K) \simeq \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(K^{ur}/K)$

$$\mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(K^{ur}/K) \simeq \mathcal{O}_K^\times \times \hat{\mathbb{Z}}.$$

$a = u\pi^v$:

(1) $\psi_\pi(a)|_{K^{ur}} = \mathrm{Frob}_K^v$

(2) $\psi_\pi(a)|_{K_\pi} = u^{-1}$

(inverse? Othw. may dep. on π).