Math 31BH, Winter 2017

Honors Multivariable Calculus, HW 3

Due Monday January 30th 2017 at 2:15 PM in Eric's box

Section 1.5 (p. 101): 11, 12, 18

Section 1.6 (p. 118): 1, 2, 6 (you may use A below for Exercise 1.6.2)

Problem A. Fix a point $y \in \mathbb{R}^n$ once and for all, and define the function $f : \mathbb{R}^n \to \mathbb{R}$ to be the distance from that point: $f(x) = d(x, y) = \|x - y\|$.

(a) Verify that $|f(x) - f(x')| \leq \|x - x'\|$ for all $x, x' \in \mathbb{R}^n$.

(b) Conclude that f is uniformly continuous.

(c) What is the preimage $f^{-1}((-\infty, r))$? Give a geometric description.

Problem B. Let $X \subset \mathbb{R}^n$ be an open subset, and let $f : X \to \mathbb{R}^m$ be an arbitrary function defined on it.

(a) Show that f is continuous if and only if the preimage

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

is open for every open subset $V \subset \mathbb{R}^m$.

(b) Check that $f^{-1}(\mathbb{R}^m - V) = X - f^{-1}(V)$ for all subsets $V \subset \mathbb{R}^m$.

(c) Assuming X is closed (instead of open) use (b) to prove the analogue of (a) for all *closed* subsets $V \subset \mathbb{R}^m$.

Problem C. Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Let f' denote its derivative.

(a) Suppose $f(a) = f(b)$. Show that $f'(c) = 0$ for some $c \in (a, b)$.

(b) Use (a) to prove the more general statement (the so-called mean value theorem) that there always exists a point $c \in (a, b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(`Hint`: Apply (a) to the function $F(x) := f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$.)

(c) Deduce that sin, cos, and arctan, are uniformly continuous on $\mathbb{R}$.