Math 31BH, Winter 2017

Honors Multivariable Calculus, HW 4

Due Wednesday February 8th 2017 at 2:15 PM in Eric's box

Section 1.7 (p. 134): 1, 2, 5, 6, 7, 11, 14 (generalize to g with values in $\mathbb{R}^m$)

Problem A. Recall that the *supremum* of a non-empty bounded set $A \subset \mathbb{R}$ is defined as the least upper bound for A. It is denoted $\sup(A)$. Similarly, its *infimum* $\inf(A)$ is the greatest lower bound.

(a) Show that there is an increasing sequence $\{a_m\}_{m=1}^{\infty}$ in A converging to $\sup(A)$. (Here by *increasing* we mean that $a_1 \leq a_2 \leq \cdots \leq a_m \leq \cdots$)

(b) Conversely, suppose $\{b_m\}_{m=1}^{\infty}$ is a bounded increasing sequence in $\mathbb{R}$. Show that it converges to $\sup(B)$, where $B := \{b_1, b_2, \ldots, b_m, \ldots\}$.

(c) Let $\{c_m\}_{m=1}^{\infty}$ be an arbitrary bounded sequence in $\mathbb{R}$. Define another sequence $\{d_m\}_{m=1}^{\infty}$ by letting $d_m := \inf\{c_m, c_{m+1}, \ldots\}$. Deduce from (b) that d_m converges as $m \to \infty$. (The limit is denoted $\liminf_{m\to\infty} c_m$ and is called the *limit inferior* of the original sequence $\{c_m\}_{m=1}^{\infty}$.)

Problem B. Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by the formula $f(x,y) = \frac{xy^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$, and at the origin declare that $f(0,0) = 0$.

(a) Show f is continuous, and that it vanishes on the axes $x = 0$ and $y = 0$.

(b) Explain why the two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point $(x,y) \in \mathbb{R}^2$ and compute them explicitly. Are they continuous at $(0,0)$?

(c) Find the directional derivative of f at $(0,0)$ in the direction $(1,1)$, if it exists, and conclude that f is *not* differentiable at the origin.