Math 31BH, Winter 2017

Honors Multivariable Calculus, HW 5

Due Monday February 13th 2017 at 2:15 PM in Eric's box

Section 1.7 (p. 134): 16(a), 18, 19

Section 1.8 (p. 143): 1

Section 1.9 (p. 151): 1, 2(a), 3

Problem A. Fix a point $y \in \mathbb{R}^n$ and define a function $f : \mathbb{R}^n \to \mathbb{R}$ to be the distance $f(x) = \|x - y\|$ (cf. Problem A on Homework 3).

(a) Explain why the partial derivatives $\frac{\partial f}{\partial x_j}$ do not exist at the point y.

(b) Show that all partial derivatives of f exist *away* from y, and that for all points $x \neq y$ the gradient of f is given by the formula

$$\nabla f(x) = \frac{x - y}{\|x - y\|}.$$

(c) Verify that f is continuously differentiable on the complement $\mathbb{R}^n - \{y\}$.

Problem B. Let $U \subset \mathbb{R}^n$ be an open convex[1] set, and $f : U \to \mathbb{R}$ a continuous function. Fix $a, b \in U$ and assume $D_{b-a}f$ exists at all points in U.

(a) Use the mean value theorem (in one variable) to prove the existence of a point $c \in U$ where

$$D_{b-a}f(c) = f(b) - f(a).$$

Can you say more about the location of c?

(b) Suppose f is differentiable on U with bounded gradient. (That is, there is a constant M such that $\|\nabla f(c)\| \leq M$ for all $c \in U$.) Show that f is uniformly continuous on U.

(c) Generalize (b) to vector-valued differentiable functions $f : U \to \mathbb{R}^m$ with bounded total derivative. (That is $\|Df(c)\| \leq M$ for all $c \in U$, where $\|\cdot\|$ here means the matrix norm.)

[1] I.e., the line segment joining any two points in U is contained in U; balls are convex.