Math 31BH, Winter 2017

Honors Multivariable Calculus, HW 8

Due Monday March 6th 2017 at 2:15 PM in Eric's box

Section 3.3 (p. 323): 3, 4(a), 6

Section 3.4 (p. 331): 1, 3, 6, 9

Problem A. Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $v \in \mathbb{R}$ be an arbitrary number lying between $f(a)$ and $f(b)$. In this problem we show that there exists a point $c \in [a, b]$ for which $f(c) = v$. (This is known as the *intermediate value theorem.*)

(a) First reduce to the case where $f(a) < v < f(b)$. (`Hint`: Consider $-f$.)

(b) Explain why $S := \{x \in [a, b] : f(x) < v\}$ is non-empty and bounded; then show that $c := \sup(S)$ satisfies the inequality $f(c) \leq v$.

(c) Prove that $f(c) = v$ by contradiction. (`Hint`: If $f(c) < v$ let $\epsilon = v - f(c)$ and get a δ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Deduce that all such $x \in [a, b] \cap (c - \delta, c + \delta)$ lie in S, and infer that $c = b$.)

(d) Conclude that the range is the interval $f([a, b]) = [\min f, \max f]$.

Problem B. Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions, and assume that g is non-negative (meaning that $g(t) \geq 0$ for all $t \in [a, b]$).

(a) Using part (d) of Problem A above, show the existence of a point $c \in [a, b]$ with the property

$$\int_a^b f(t)g(t)dt = f(c) \cdot \int_a^b g(t)dt.$$

(`Hint`: Multiply $\min(f) \leq f(t) \leq \max(f)$ by $g(t)$ and integrate.) Does this still hold if we instead assume g is non-positive?

(b) Taking $g(t) = 1$ deduce the *mean-value theorem for integrals*: For some c,

$$\frac{1}{b-a} \cdot \int_a^b f(t)dt = f(c).$$

(c) Give an example illustrating the conclusion in (a) *fails* if we do not impose conditions on g. (`Hint`: Take $f(t) = g(t) = t$ on $[-1, +1]$.)