Math 31BH, Winter 2017

Honors Multivariable Calculus, HW 9

Due Thursday March 16th 2017 at 10:15 AM in Eric's box

Section 2.10 (p. 275): 1, 5

Section 3.6 (p. 348): 1, 2, 5

Problem A. In the proof of the *inverse function theorem*[1] we appealed to the following result, which we will show in this exercise:

Lemma 1. *Let $f : U \to \mathbb{R}$ be a function defined on an open subset $U \subset \mathbb{R}^n$ containing a. Then f is differentiable at a if and only if there exists a function $\Phi : U \to \mathbb{R}^n$ which is continuous at a such that*

$$f(x) - f(a) = \Phi(x) \bullet (x - a), \qquad \forall x \in U. \tag{1}$$

Moreover, any such Φ necessarily satisfies the relation $\Phi(a) = \nabla f(a)$.

(a) First suppose f is differentiable at a. I.e., there is a function $\epsilon_a(x)$ defined in a neighborhood of a, and continuous at a, such that $\epsilon_a(a) = 0$ and

$$f(x) = f(a) + \nabla f(a) \bullet (x - a) + \epsilon_a(x) \cdot \|x - a\|$$

holds for x in a neighborhood of a. Define $\Phi(x)$ by the formula

$$\Phi(x) := \nabla f(a) + \epsilon_a(x) \cdot \frac{x - a}{\|x - a\|}$$

for $x \neq a$, and declare that $\Phi(a) = \nabla f(a)$. Check this Φ does the job.

(b) Now suppose we have a continuous $\Phi(x) = (\phi_1(x), \dots, \phi_n(x))$ such that (1) holds. Verify that $\frac{\partial f}{\partial x_j}(a)$ exists for all j and equals $\phi_j(a)$. Conclude that $\Phi(a) = \nabla f(a)$.

(c) Continuing with the setup in (b) show the inequality

$$\frac{|f(x) - f(a) - \nabla f(a) \bullet (x - a)|}{\|x - a\|} \leq \|\Phi(x) - \Phi(a)\|$$

for all $x \in U - \{a\}$. Infer that f is differentiable at a.

[1] Its statement is recalled in Problem C below.

(d) Extend Lemma 1 to vector-valued functions $f : U \to \mathbb{R}^m$.

Problem B. Let $K \subset \mathbb{R}^n$ be a compact subset.

(a) Show that $\sup\{\|x - k\| : k \in K\}$ is *finite* for all $x \in \mathbb{R}^n$. (`Hint`: Use the result of Problem A, part (b), on HW3, and Theorem 1.6.9 in the book.)

(b) Let $d(x, K) := \inf\{\|x - k\| : k \in K\}$. Prove that $d(x, K) = 0 \iff x \in K$.

(c) Show that $\forall x, y \in \mathbb{R}^n$ the following inequality holds:

$$|d(x, K) - d(y, K)| \leq \|x - y\|.$$

Conclude that the function $x \mapsto d(x, K)$ is uniformly continuous.

Problem C. The version of the inverse function theorem proved in class is (apart from local injectivity) the following:

Theorem 1. *Let $f : U \to \mathbb{R}^n$ be an injective function of class $\mathcal{C}^1$ defined on an open subset $U \subset \mathbb{R}^n$. Suppose its Jacobian matrix $Jf(x)$ is invertible $\forall x \in U$. (I.e., has nonzero determinant). Then the image $f(U)$ is an open subset of $\mathbb{R}^n$, and the inverse function $f^{-1} : f(U) \to \mathbb{R}^n$ is of class $\mathcal{C}^1$.*

– Does this result extend to functions of class $\mathcal{C}^p$ for every $p \leq \infty$? In other words, if f is of class $\mathcal{C}^p$ does that guarantee the inverse f^{-1} is also of class $\mathcal{C}^p$? (`Hint`: Use the formula $A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A)$ from linear algebra.)

Problem D. In this problem we will show that every symmetric $n \times n$ matrix A (with entries in $\mathbb{R}$) *has* at least one real eigenvalue. (This is the key to proving the full *spectral theorem* that $\mathbb{R}^n$ admits an orthonormal eigenbasis for A.)

(a) We introduce the quadratic form $Q_A(x) = xAx^T$ and the *Rayleigh* quotient $R_A(x) = \frac{Q_A(x)}{\|x\|^2}$ (defined for $x \neq 0$). Show that $|R_A(x)| \leq \|A\|$ for all nonzero x. (`Hint`: Cauchy-Schwarz.)

(b) Verify the formulas $\nabla Q_A(x) = 2xA$ and $\nabla R_A(x) = 2x(A - R_A(x)I)$.

(c) Explain why $R_A(x)$ has a minimum and a maximum on $\mathbb{R}^n - \{0\}$. (`Hint`: $R_A(cx) = R_A(x)$ for all $c \neq 0$; so consider R_A on the unit circle.)

(d) Let x_0 be a local extremum for $R_A(x)$. Deduce from (b) that $R_A(x_0)$ is an eigenvalue for A (with eigenvector x_0^T).

(e) Pinpoint *where* you used that A is symmetric.