MATH 31BH, HONORS MULTIVARIABLE CALCULUS, MIDTERM 1

Wednesday, February 1st, 2017, 1-1:50pm, MANDE B-150

- *Your Name*: SOLUTIONS
- *ID Number*:
- *Section (circle)*: A01 (7:00 PM) A02 (8:00 PM)

All answers must be fully justified

Problem #	Points (out of 10)
1	
2	
3	
4	
5	
Total (out of 50):	

Problem 1. Draw the following subsets in the plane $\mathbb{R}^2$, and in each case indicate whether the subset is open, closed, or neither.

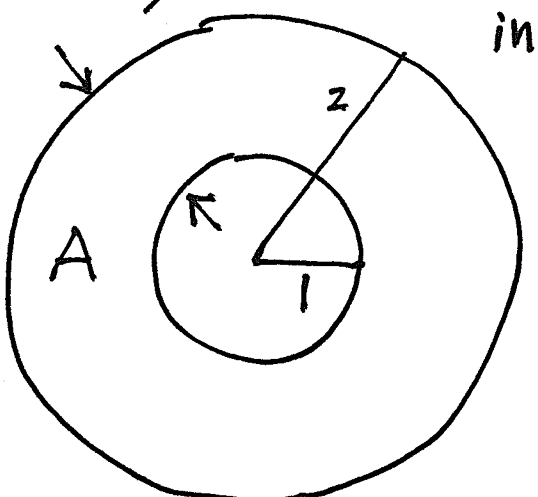
(a) $A = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$

(b) $B = \{(x, y) \in \mathbb{R}^2 \colon |x| < 1 \text{ and } |y| \leq 4\}$

(c) $C = \{(x, y) \in \mathbb{R}^2 : 1 \leq x + y \leq 4\}$

(a)

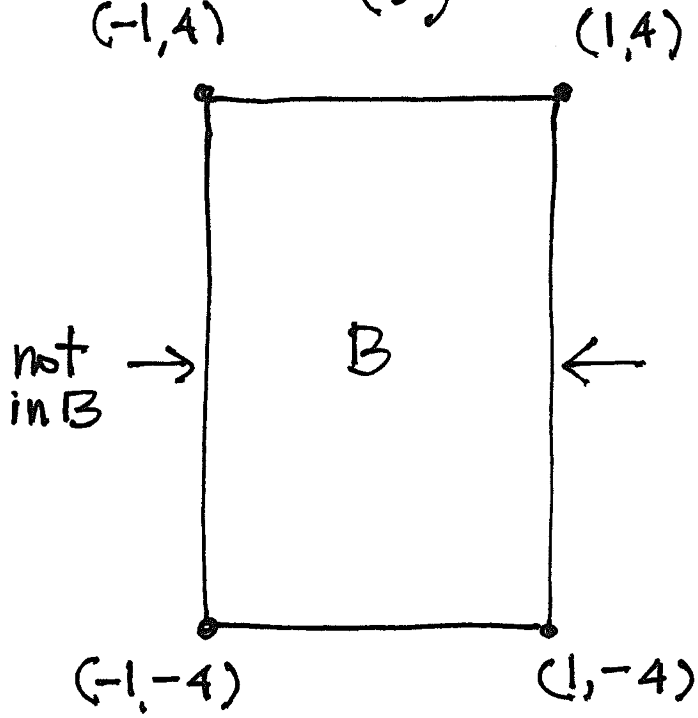
- no boundary points included in A.



"annulus"

A is <u>open</u>.

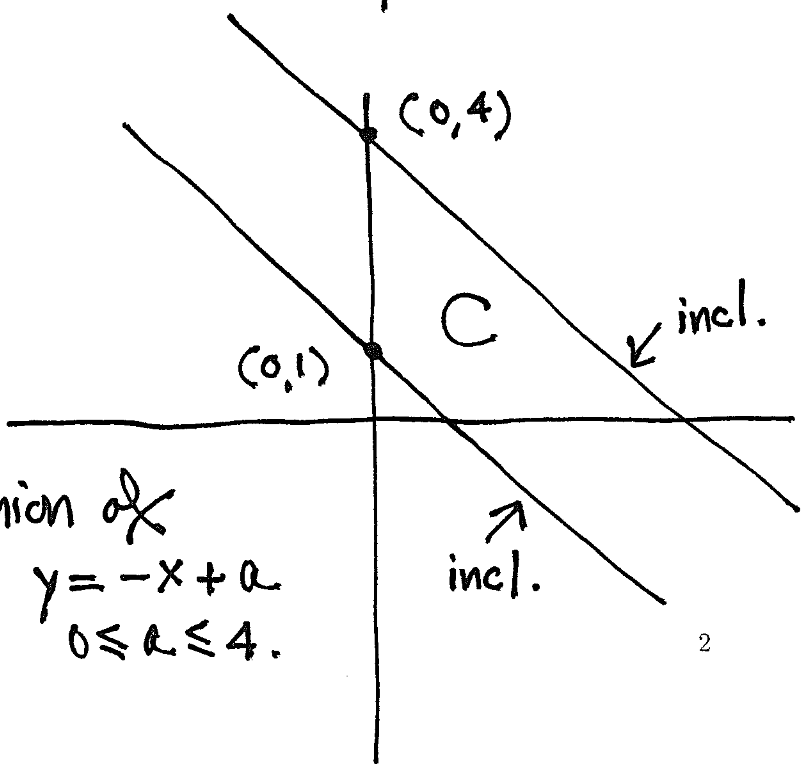
(b)



"vertical half-open rectangle"

B is <u>neither</u> open nor closed.

(c)



C: union of lines $y = -x + a$ with $0 \leq a \leq 4$.

C "infinite strip of slope -1"

C is <u>closed</u>

Problem 2. Mark each statement *true* (T) of *false* (F). Justify your answers.

T (a) The interior A° of a subset $A \subset \mathbb{R}^n$ is always open.

T (b) The closure of the set of rational numbers $\mathbb{Q}$ is all of $\mathbb{R}$.

F (c) The boundary of a closed subset $A \subset \mathbb{R}^n$ is always empty.

F (d) A and A° have the same closure for any $A \subset \mathbb{R}^n$.

T (e) The boundary ∂A is closed for any subset $A \subset \mathbb{R}^n$.

(a) TRUE, $\overset{\circ}{A}$ is the largest open subset of A.

(b) TRUE, if $x \in \mathbb{R}$ any open interval $(x-\varepsilon, x+\varepsilon)$ contains rational numbers. I.e., $x \in \overline{\mathbb{Q}}$.

(c) FALSE, take $A = [0,1] \subseteq \mathbb{R}$ (closed), $\partial A = \{0,1\}$ is non-empty.

(d) FALSE, take $\mathbb{Q} \subseteq \mathbb{R}$. Has no interior points — any $(x-\varepsilon, x+\varepsilon)$ contains irrational numbers. i.e.,

$$\overset{\circ}{\mathbb{Q}} = \emptyset \quad (\text{closed, so } \overline{\emptyset} = \emptyset)$$

However, $\overline{\mathbb{Q}} = \mathbb{R}$ (cf. (b))

(e) TRUE,

$$\partial A \overset{\text{def.}}{=\!=} \overline{A} - \overset{\circ}{A} = \underset{\text{closed}}{\overline{A}} \cap \underset{\text{closed (cf. (a))}}{(\overset{\circ}{A})^c}$$

- finite intersections of closed subsets is closed.

Problem 3. Determine if the limits below exist or not. If the limit does exist, find it.

(a) $\lim_{(x,y)\to(3,-1)}(2x+y^3, x^2-y)$ $= (5, 10)$

(b) $\lim_{(x,y)\to(0,0)} \frac{x}{y}$ not exist.

(c) $\lim_{x\to 0}(\frac{\sin x}{x}, \arctan\frac{1}{x})$ not exist.

(In the last question $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is the inverse function of tan.)

(a) Polynomials are continuous, so the limit exists and equals the value of the function at $(3,-1)$:

$$\left(2\cdot 3 + (-1)^3,\ 3^2 - (-1)\right) = \underline{(5, 10)}.$$

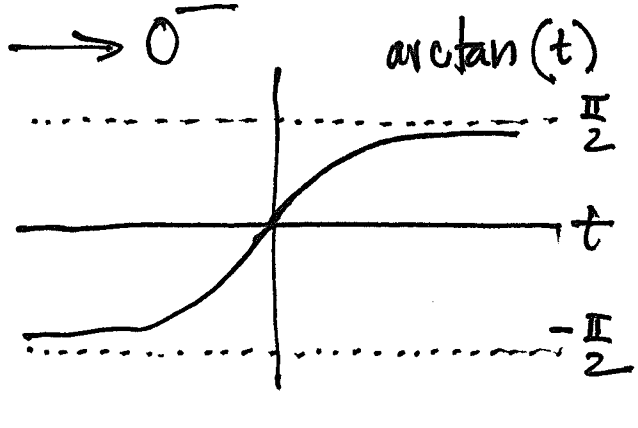
(b) Limit does not exist: The function $f(x,y) = \frac{x}{y}$ is constant $= \alpha$ on the line $y = \frac{1}{\alpha}x$, so

$$\lim_{x\to 0} f(x, \tfrac{1}{\alpha}x) = \alpha$$ depends on the inverse slope $\frac{1}{\alpha}$.

(c) Limit does not exist: $\lim_{x\to 0}\frac{\sin x}{x} = \begin{smallmatrix}\sin'\\ \cos\end{smallmatrix}(0) = 1$,

– however:

$$\underbrace{\arctan\frac{1}{x}}_{\text{– has no limit as } x\to 0 \text{ (dep. on sign of } x)} \begin{matrix} \nearrow +\frac{\pi}{2} \text{ as } x \to 0^+ \\ \searrow -\frac{\pi}{2} \text{ as } x \to 0^- \end{matrix}$$



Problem 4. Consider some sequence of vectors in $\mathbb{R}^n$,

$$\underline{x}_1,\ \underline{x}_2,\ \underline{x}_3,\ \ldots\ldots\ \underline{x}_m,\ \ldots$$

(a) Prove that if the sequence converges it must be bounded[1]

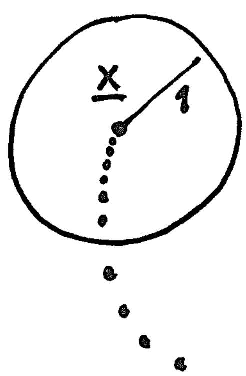
(b) Give an example of a bounded sequence in $\mathbb{R}$ which does *not* converge.

(c) Does your sequence from (b) admit a convergent subsequence?

(a) Say $\underline{x}_m \to \underline{x}$ as $m \to \infty$. In particular, all but finitely many terms lie in the ball $B_1(\underline{x})$.

↑ call them $\underline{x}_1, \ldots, \underline{x}_M$

(i.e., $\|\underline{x}_m - \underline{x}\| < 1$ for $m > M$)

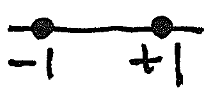


- Can take any R which is greater than $1 + \|\underline{x}\|$ and greater than all $\|\underline{x}_1\|, \ldots, \|\underline{x}_M\|$

$$\begin{cases} \bullet \text{ all } \underline{x}_1, \ldots, \underline{x}_M \text{ lie in } B_R(\underline{0}) \\ \bullet \text{ For } m > M,\ \|\underline{x}_m\| \le \underbrace{\|\underline{x}_m - \underline{x}\|}_{<1} + \|\underline{x}\| < 1 + \|x\| < R. \end{cases}$$

i.e. $\underline{x}_m \in B_R(\underline{0})$.

(b) $x_m = (-1)^m = \begin{cases} +1 & m \text{ even} \\ -1 & m \text{ odd.} \end{cases}$



jumps between two points; ∞ many times. **not** convergent but bounded: $x_m \in [-1, 1] = \overline{B_1(0)} \subset B_2(0)$.

(c) Does have a conv. subsequence (any bounded seq. does)

- can take all "even" (or "odd") terms:

$1 = x_2 = x_4 = x_6 = \cdots$ constant $\Rightarrow$ convergent.

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[1] I.e., all its terms $\underline{x}_m$ lie in some sufficiently large ball $B_R(\underline{0}) \subset \mathbb{R}^n$.

Problem 5. Decide which of the functions below are *uniformly* continuous. In each case explain why or why not.

✓ (a) $f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = \sin(x)$

%. (b) $g : (0, \infty) \longrightarrow \mathbb{R}, \quad g(x) = \frac{1}{x}$

✓ (c) $h : [0, \infty) \longrightarrow \mathbb{R}, \quad h(x) = \sqrt{x}$

(a) $\sin(x)$ is unifomly continuous on $\mathbb{R}$ since $\sin' = \cos$ is bounded: $|\cos(x)| \leqslant 1$, all $x \in \mathbb{R}$.

— use the mean-value thm. (cf. HW3): $\frac{f(a)-f(b)}{a-b} = f'(c)$.

$\Rightarrow |f(a)-f(b)| \leqslant |a-b|$ (can take $\delta = \varepsilon$).

(b) $\frac{1}{x}$ not unf. cont. on $(0,\infty)$.

Suppose it is. Then $\exists \delta$:

$$\left|\frac{1}{x} - \frac{1}{y}\right| < 1 \text{ whenever } |x-y| < \delta.$$

Let $x \in (0,\infty)$ be arbitrary, and put $y = x + \frac{\delta}{2}$. Infer:

$$\underbrace{\left|\frac{1}{x} - \frac{1}{x+\frac{\delta}{2}}\right|}_{= \frac{\delta/2}{x(x+\delta/2)}} < 1$$

$= \frac{\delta/2}{x(x+\delta/2)}$; becomes arbitrarily large ($\infty \geqslant 1$) when x gets closer to 0 (from the right).

— contradiction.

(c) $\sqrt{x}$ is unif. cont. on $[0,\infty)$: (1) It's u.c. on the compact $[0,1]$.

(2) It's u.c. on $[1,\infty)$ since its $h'(x) = \frac{1}{2\sqrt{x}} \leqslant \frac{1}{2}$ is bounded here; use MVT as in (a).

—— combine (1) & (2).