MATH 31BH, HONORS MULTIVARIABLE CALCULUS, MIDTERM 2

Wednesday, March 1st, 2017, 1-1:50pm, MANDE B-150

- *Your Name*: SOLUTIONS.
- *ID Number*:
- *Section (circle)*: A01 (7:00 PM) A02 (8:00 PM)

All answers must be fully justified. You have an *extra page* at the very end of the exam.

Problem #	Points (out of 10)
1	
2	
3	
4	
5	

Total (out of 50):

Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = x^2 \sin(y)$.

(a) Find its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at any point (x, y).

(b) Write down the gradient $\nabla f(x, y)$.

(c) Compute the directional derivative at $(\sqrt{2}, \frac{\pi}{4})$ in the direction $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(d) Give an equation for the tangent plane to the graph of f at $(\sqrt{2}, \frac{\pi}{4})$.

(Hint: $\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.)

(a) $\frac{\partial f}{\partial x}(x,y) = 2x\sin(y)$ and $\frac{\partial f}{\partial y}(x,y) = x^2\cos(y)$.

(b) $\nabla f(x,y) = (2x\sin(y), x^2\cos(y))$.

(c) f clearly (continuously) differentiable, so with $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$:

$$D_v f(\sqrt{2}, \tfrac{\pi}{4}) = \nabla f(\sqrt{2}, \tfrac{\pi}{4}) \bullet v$$
$$= (2, \sqrt{2}) \bullet (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = \underline{\underline{1 + \sqrt{2}}}.$$

(d) The equation is, in coordinates (x, y, z): $a = (\sqrt{2}, \frac{\pi}{4})$.

$$z = f(a) + \frac{\partial f}{\partial x}(a)(x - \sqrt{2}) + \frac{\partial f}{\partial y}(a)(y - \tfrac{\pi}{4})$$
$$= \sqrt{2} + 2(x - \sqrt{2}) + \sqrt{2}(y - \tfrac{\pi}{4}).$$
$$= \underline{\underline{2x + \sqrt{2}y + -(1 + \tfrac{\pi}{4})\sqrt{2}}}.$$

Problem 2. Mark each statement *true* (T) or *false* (F). Justify your answers: If true briefly explain why, if false give a counterexample.

F (a) A function is differentiable if all partial derivatives exist at all points.

F (b) "Continuously differentiable" means continuous and differentiable.

F (c) The Jacobian matrix J of a function $\mathbb{R}^2 \to \mathbb{R}^2$ is symmetric ($J^T = J$).

T (d) A function f with Jacobian matrix $Jf(x, y) = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}$ is differentiable.

T (e) A differentiable function is necessarily continuous.

(a) FALSE: $f(x,y) = \dfrac{xy^2}{x^2+y^2}$ counterex., cf. Problem B on HW4.

(b) FALSE: "cont. diff." means all $\dfrac{\partial f}{\partial x_j}$ (exist and) are continuous.

– recall: $x^2 \sin(\frac{1}{x})$ is diff, but not cont. diff.

(c) FALSE: Take any non-sym. matrix A and consider
$$f : \mathbb{R}^2 \to \mathbb{R}^2,\quad x \mapsto Ax \quad ; \text{ has Jacobian } J = A.$$
– to be concrete: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ gives $f(x,y) = y$.

(d) TRUE: If $f : \mathbb{R}^2 \to \mathbb{R}^2$ has $Jf(x,y) = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}$,
– it's visibly continuously differentiable $\Rightarrow$ diff.
$\left(\dfrac{\partial f}{\partial x} = y \text{ and } \dfrac{\partial f}{\partial y} = x : \text{cont.}\right)$
[Think of $f(x,y) = (xy, x+y)$.]

(e) TRUE:
$$f(a+h) - f(a) = \frac{f(a+h) - f(a) - A(h)}{\|h\|} \|h\| + A(h) \to 0.$$

Problem 3. Below J denotes the Jacobian matrix, and ∇ the gradient.

(a) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (\cos(x + y), \sin(x - y))$. Find $Jf(x, y)$.

(b) Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x, y) = \sin(x^2 + y^3)$. Find $\nabla g(x, y)$.

(c) Define $h : \mathbb{R}^3 \to \mathbb{R}$ by $h(x, y, z) = e^{\cos(xy)+\sin(yz)}$. Find $\nabla h(x, y, z)$.

(Hint: Recall the formulas $\sin'(t) = \cos(t)$ and $\cos'(t) = -\sin(t)$.)

(a) $$Jf(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -\sin(x+y) & -\sin(x+y) \\ \cos(x-y) & -\cos(x-y) \end{pmatrix}$$

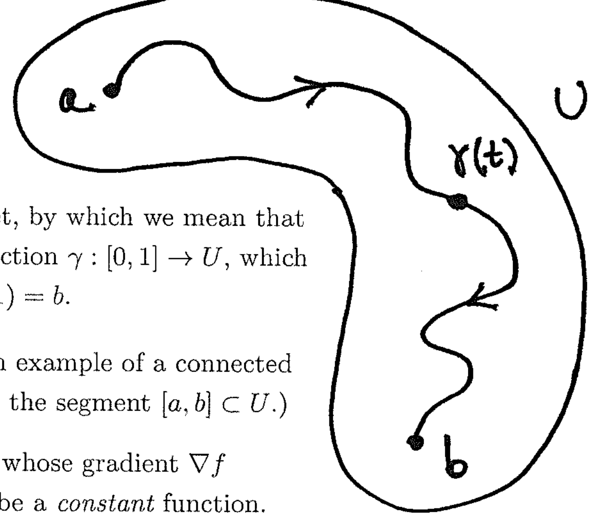
(b) $$\nabla g(x,y) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) = \left(2x\cos(x^2+y^3),\ 3y^2\cos(x^2+y^3) \right)$$

(c) $$\nabla h(x,y,z) = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z} \right) =$$

$$\underline{\underline{-y\sin(xy)\, e^{\cos(xy)+\sin(yz)}}}$$

$$\underline{\underline{y\cos(yz)\, e^{\cos(xy)+\sin(yz)}}}$$

$$\underline{\underline{\left(-x\sin(xy) + z\cos(yz)\right) e^{\cos(xy)+\sin(yz)}}}$$



Problem 4. Let $U \subset \mathbb{R}^n$ be a *connected* open subset, by which we mean that for any two points $a, b \in U$ there is a continuous function $\gamma : [0,1] \to U$, which is differentiable on $(0,1)$, such that $\gamma(0) = a$ and $\gamma(1) = b$.

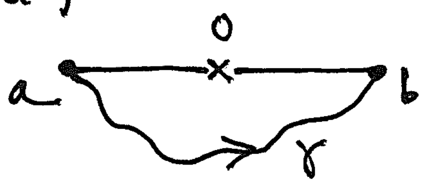
(a) Verify that convex $\Longrightarrow$ connected; then give an example of a connected *non*-convex set. (`Hint`: Recall "convex" means the segment $[a,b] \subset U$.)

(b) Suppose $f : U \to \mathbb{R}$ is a differentiable function whose gradient ∇f vanishes identically[1] in U. Prove that f must be a *constant* function.

(c) Give an example showing that the conclusion in (b) is <u>false</u> when U is *not* connected. (`Hint`: Take U to be the union of two disjoint open balls.)

(a) If U is <u>convex</u>, may take $\gamma(t) = a + t(b-a)$ which parametrizes the <u>line segment</u> $[a,b] \subseteq U$.

$\mathbb{R}^n - \{0\}$ is <u>not</u> convex,
but it <u>is</u> connected:



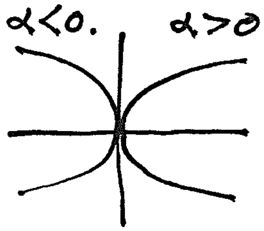
(b) Let $a, b \in U$ be arbitrary. Must show $f(a) = f(b)$.
– By the Mean-Value Thm., (in one var.) applied to $g(t) := f(\gamma(t))$ (well-defined!) $\gamma(t) \in U$

$$f(b) - f(a) = \frac{g(1) - g(0)}{1 - 0} \overset{\exists s \in (0,1):}{=} g'(s) \underset{\text{C.R.}}{=} \underbrace{\nabla f(\gamma(s))}_{0 \text{ by assumption.}} \bullet \gamma'(s) = 0. \checkmark$$

(c) $U = B_1 \cup B_2$ disj. open balls.
– Let $f(x) = \begin{cases} 1 & x \in B_1 \\ 2 & x \in B_2 \end{cases}$. <u>Not</u> constant, <u>but</u> $\nabla f \equiv 0$.

[1] Meaning that $\nabla f(a)$ is the zero-vector for all points $a \in U$.

Problem 5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by the formula $f(x, y) = \frac{xy^2}{x^2+y^4}$ at points $(x, y) \neq (0, 0)$, and declare that $f(0, 0) = 0$.



(a) Show that f is *not* continuous at $(0, 0)$. (Hint: Substitute $y^2 = \alpha x$.)

(b) Verify that the directional derivatives of f *do* exist at all points and in *every* direction.

(c) For $v = (a, b)$ compute $D_v f(0, 0)$ explicitly as a function of a and b.

(a) $f(x,y) = \dfrac{x(\alpha x)}{x^2+\alpha^2 x^2} = \dfrac{\alpha}{1+\alpha^2}$, (constant) so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

along $y^2 = \alpha x$.

$\Rightarrow f$ not cont. at $(0,0)$.

(b) Away from $(0,0)$ the function f is clearly smooth, so the point of interest is the origin $(0,0)$. Let $v=(a,b)$ nonzero (but arbitrary).

$$D_v f(0,0) = \lim_{t\to 0} \frac{f(ta,tb)-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{1}{t}\cdot\frac{(ta)(tb)^2}{(ta)^2+(tb)^4} \qquad (f(0,0)=0)$$

$$= \lim_{t\to 0} \frac{ab^2}{a^2+t^2b^4}$$ exists, and equals:

$$= \frac{ab^2}{a^2}$$

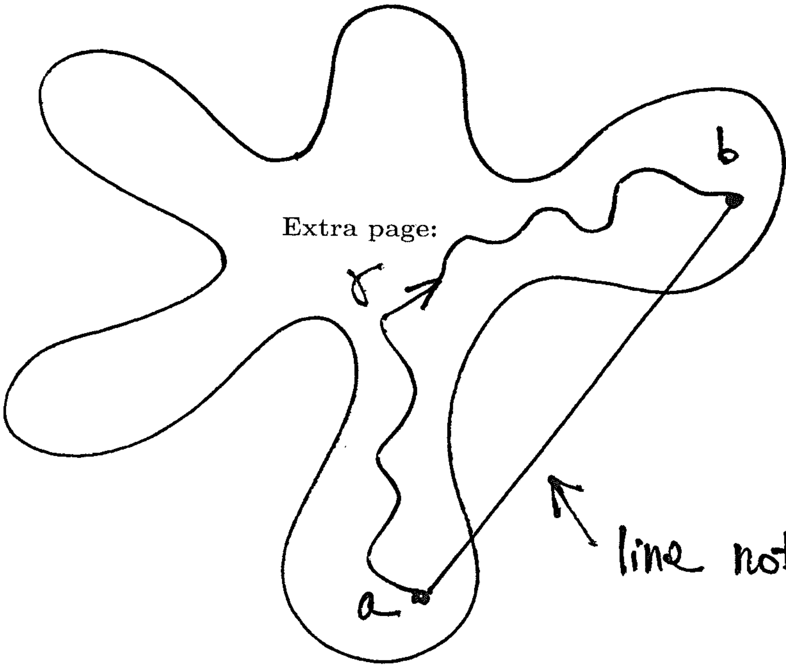
$$= \frac{b^2}{a} \text{ if } a\neq 0$$

(which is the answer to part (c))

☆ When $a=0$ $(b\neq 0)$:

$D_v f(0,0) = 0.$

Extra page:



U connected, not convex.

line not within U.