TOPOLOGY REVIEW #17: CLASSIFICATION OF COVERING SPACES

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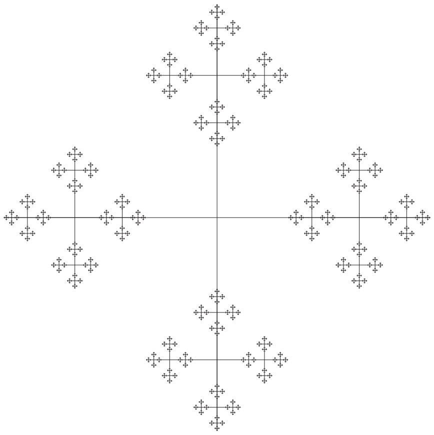

1. INTRODUCTION

We like covering spaces, because they're cool, and help us understand the properties and structure of a space. One of the most important things we can do, then, is attempt to classify covering spaces, and also determine their relationship to the fundamental group. It turns out that, given a space B (with some modest connectivity hypotheses), having a simply connected cover E induces a correspondence between subgroups of the fundamental group $\pi_1(B)$ and their quotients with covering spaces "in between" E and B (this is called the "Galois correspondence" in analogy to the structure of subfields and field automorphism groups). In fact we can always find such a covering space if B is connected, locally path-connected, and *semilocally simply connected*—basically a space having a basis of open sets in which loops can be shrunk to a point, possibly moving outside the basis elements (we will give precise definitions and an overview of the construction in the appendix, but an understanding of the construction is not essential to our main objective of classification). This allows us to deduce a few important (and powerful) classification theorems. In the following we only consider spaces that are connected, locally path connected, and semilocally simply connected. Also, maps will be continuous unless otherwise stated.

2. THE UNIVERSAL COVERING SPACE

Before we get to the meat of the material, it would help to know what a universal covering space is.

Definition 2.1. Let B be a space (which is connected and locally path connected). We say E is a *universal covering space* if E is a covering space of B and E is simply connected. [1]

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[1] B stands for "base" an I don't know what E stands for. It doesn't mean "cover" in German as I had surmised at first; the German word for covering space is *Überlagerung* and I've got no idea what it means, though I like the fact it uses *über*, hehe. Das ist gut, ja?

As one might guess from the terminology, the "universality" of such a space has to do with categorical nonsense—any connected covering space of B is related to E by some "mapping property" and moreover, E is unique up to homeomorphism. To fully appreciate these things we need make a few more definitions, which amounts to recognizing that the ideal maps of covering spaces ("morphisms") should be more than just continuous (topological) maps, but rather maps satisfying an additional property. Let

Definition 2.2. Let B be a space and suppose $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are two covering spaces for B. Then a (continuous) map $\varphi : E_1 \to E_2$ is called a *covering homomorphism* or *morphism of covering spaces* if $p_2 \circ \varphi = p_1$ i.e. the following diagram commutes

$$\begin{tikzcd}
E_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\
& B &
\end{tikzcd}$$

If φ is invertible and its inverse is also a covering homomorphism, then φ is called a *covering isomorphism.*[2]

In particular we have already encountered the example where $E_1 = E_2$ and $p_1 = p_2$ and φ is a homeomorphism—we called that a covering transformation or deck transformation of the covering space E over B. Before continuing we need to introduce a most important theorem:

Theorem 2.3 (Lifting Criterion)**.** *Let $p : (E, e_0) \to (B, b_0)$ be a covering space. Then if $f : (Y, y_0) \to (B, b_0)$ is a map, there exists a lift $f' : (Y, y_0) \to (E, e_0)$ making the following diagram commute*

$$\begin{tikzcd}
& (E, e_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[ur, dashed, "f'"] \arrow[r, "f"] & (B, b_0)
\end{tikzcd}$$

if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$. Moreover if Y is connected, f' is the unique lift taking y_0 to e_0.

Remark 2.4. We will use the pair notation (E, e_0) to denote the space E with a distinguished base point e_0. We emphasize that keeping track of the base points is an important—many of our uniqueness statements are actually only unique up to choice of base point. Without this restriction, a map can have zillions of lifts, and chaos reigns: for example, taking $\mathbb{R}$ as the covering space over S^1 via the usual exponential map, without regard to start points, the path describing the circle itself lifts to a path traversing $[n, n+1]$ for every $n \in \mathbb{Z}$; which is very not unique.[3]

An immediate application of our lifting criterion is the following:

Theorem 2.5. *Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be covering spaces of B. Choose two base points $e_1 \in E_1$ and $e_2 \in E_2$ such that their image is a basepoint $b_0 \in B$ i.e. $b_0 = p_1(e_1) = p_2(e_2)$. Then*

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[2] This says, in the language of Categorical Nonsense, that covering spaces of X along with covering maps p form the objects of a category, and covering homomorphisms are the morphisms in this category.

[3] "Remember, you are unique. Just like everyone else"—"Individuality" poster mocking motivational posters on the door outside AP&M 2226

there exists a unique covering homomorphism $\varphi : (E_1, e_1) \to (E_2, e_2)$ *(taking* e_1 *to* e_2*) if and only if* $p_{1*}(\pi_1(E_1, e_1)) \subseteq p_{2*}(\pi_1(E_2, e_2))$*. In pretti pictures,*

$$\begin{tikzcd}
(E_1, e_1) \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & (E_2, e_2) \arrow[dl, "p_2"] \\
& (B, b_0) &
\end{tikzcd}$$

commutes.

Proof. A map $\varphi : E_1 \to E_2$ satisfying $p_1 = p_2 \circ \varphi$ means that φ is a lift of $p_1 : E_1 \to X$. Essentially the diagram in the definition of covering homomorphism is nothing but a slight variation of that in the lifting lemma, with E_1 taking place of the Y but in a more elevated position as if it were more on equal ground with E_2. Since E_1 is connected, any such lift sending e_1 to e_2 must be unique. The existence now follows because $p_{1*}(\pi_1(E_1, e_1)) \subseteq p_{2*}(\pi_1(E_2, e_2))$ is exactly the condition needed for a lift to exist. □

It turns out that such a covering homomorphism $\varphi : E_1 \to E_2$ is not just any ol' continuous map. It's covering space map (6.2 in Greenberg & Harper).

Theorem 2.6. *Let* (E_1, e_1)*,* (E_2, e_2)*,* (B, b_0)*, and the maps* p_1*,* p_2 *be as above. A covering homomorphism* $\varphi : E_1 \to E_2$ *is a covering space map.*

Proof. First we must show that φ is surjective (its continuity is given by the lifting lemma). Given $x \in E_2$, there is a path σ from the base point e_2 to x. Now push this path down to a path in B: consider the path $\tau = p_2 \circ \sigma$. This path starts at b_0 and ends at $p_2(x)$. By the lifting lemma, this path lifts to a unique path in E_1 starting at e_1 (but not necessarily ending at a point in the fiber of e_1 since τ need not be a loop). Call this lifted path $\tilde{\sigma}$. Now $p_2 \circ \varphi \circ \tilde{\sigma} = p_1 \circ \tilde{\sigma}$ by the fact that φ is a covering homomorphism. $p_1 \circ \tilde{\sigma} = \tau$ because this is how we defined $\tilde{\sigma}$—to be a lift of τ in E_1 starting at e_1. So therefore $p_2 \circ \varphi \circ \tilde{\sigma} = \tau = p_2 \circ \sigma$, which simply says that σ and $\varphi \circ \tilde{\sigma}$ are both lifts in E_2 of the same map, τ. Since $\varphi(e_1) = e_2$ we have that $\varphi \circ \tilde{\sigma}(0) = e_2 = \sigma(0)$ i.e. they have the same start point. Therefore by unique path lifting, they must be equal. In particular, their endpoints are the same, so $x = \sigma(1) = \varphi(\tilde{\sigma}(1))$, showing that x is in the image of φ.

Now to show that it satisfies the covering map property: if $x \in E_2$ then we consider a neighborhood U of the image point $p_2(x)$ which is evenly covered by p_2, and another neighborhood V of $p_2(x)$ evenly covered by p_1. Then $U \cap V$ is a neighborhood of $p_2(x)$ which is evenly covered by both maps: $p_2^{-1}(U \cap V) = p_2^{-1}(U) \cap p_2^{-1}(V)$ and $p_2^{-1}(V)$ is open; the further restriction of the local section of p_1 on some component of $p_2^{-1}(U)$, namely intersection with $p_2^{-1}(V)$ still yields a homeomorphism; similarly with p_1. So therefore, choosing the component W of $p_2^{-1}(U)$ containing x, and intersecting it with $p_2^{-1}(V)$ we have $\varphi^{-1}(W \cap p_2^{-1}(V)) = \varphi^{-1}(W) \cap p_1^{-1}(V)$ since $p_2 \circ \varphi = p_1$. $\varphi^{-1}(W) \cap p_1^{-1}(V)$ is a disjoint union of open sets since $p_1^{-1}(V)$ is; and restricting to one of the components N we have that $p_1|_N$ is a homeomorphism on it, so that $p_1|_N = p_2 \circ \varphi|_N$. But since $\varphi(N) \subseteq W$ by hypothesis, $p_1|_N = p_2|_W \circ \varphi|_N$; since $p_2|_W$ is a homeomorphism (W was chosen as a sheet in $p_2^{-1}(U)$), this implies that $\varphi|_N$ is as well. □

Now we are ready to justify the name "universal covering space":

Theorem 2.7. *If* E *is a simply connected covering space for* B *with covering map* p*, and* $p' : X \to B$ *is another connected covering space, choosing base points* $e_0 \in E$*,* $x_0 \in X$*, and* $b_0 \in B$*, there exists a unique covering homomorphism* $\varphi : (E, e_0) \to (X, x_0)$ *(i.e. there is a unique continuous* φ *map out of* E *to* X *taking* e_0 *to* x_0 *and satisfying* $p = p' \circ \varphi$*)*

The diagram notation we use to express this yet another variation from that in the lifting lemma, now elevating E further to show that E somehow lies "above" X in a hierarchy:

$$\begin{tikzcd}
(E, e_0) \arrow[dd, "p"'] \arrow[dr, "\varphi"] & \\
& (X, x_0) \arrow[dl, "p'"] \\
(B, b_0) &
\end{tikzcd}$$

Proof. Taking $p_1 = p$ and $p_2 = p'$, since $\pi_1(E, e_0) = 0$ it is clear that $p_1(\pi_1(E, e_0)) \subseteq p_2(\pi_1(X, x_0))$. So the result follows by the above. $\square$

We observe again that the map $\varphi : E \to X$ is also a covering space map.

Corollary 2.8. *A universal covering space is unique up to homeomorphism (actually, covering isomorphism).*

Proof. [4] Let (E_1, e_1) and (E_2, e_2) be two simply connected covering spaces with respective maps p_1 and p_2. Then there is a map $\varphi : (E_1, e_1) \to (E_2, e_2)$ such that $p_1 = p_2 \circ \varphi$ and a map $\psi : (E_2, e_2) \to (E_1, e_1)$ such that $p_2 = p_1 \circ \psi$. So this says $\psi(\varphi(e_1)) = e_1$, $p_1 = p_1 \circ (\psi \circ \varphi)$. and we have the following commutative diagram:

$$\begin{tikzcd}
(E_1, e_1) \arrow[r, "\varphi"] \arrow[dr, "p_1"'] & (E_2, e_2) \arrow[r, "\psi"] \arrow[d, "p_2"] & (E_1, e_1) \arrow[dl, "p_1"] \\
& (B, b_0) &
\end{tikzcd}$$

But the identity map Id_{E_1} satisfies $Id_{E_1}(e_1) = e_1$ and $p_1 = p_1 \circ Id_{E_1}$ so we must have, by uniqueness of lifts, that $\psi \circ \varphi = Id_{E_1}$. Similarly, $\varphi(\psi(e_2)) = e_2$ and $p_2 = p_2 \circ (\varphi \circ \psi)$ with the following pretty picture

$$\begin{tikzcd}
(E_2, e_2) \arrow[r, "\psi"] \arrow[dr, "p_2"'] & (E_1, e_1) \arrow[r, "\varphi"] \arrow[d, "p_1"] & (E_2, e_2) \arrow[dl, "p_2"] \\
& (B, b_0) &
\end{tikzcd}$$

and a similar argument with uniqueness of lifts says $\varphi \circ \psi = Id_{E_2}$. This shows that $\psi = \varphi^{-1}$. Since all the maps in question are continuous (and compositions and so forth preserve continuity), φ and ψ are homeomorphisms. $\square$

This does it for uniqueness of the universal covering space; the existence of such a space, though important, is a construction relegated to the appendix. We will assume the hypotheses of the construction—namely all our spaces in question are connected, locally-path connected and semilocally simply connected, in order to ensure the existence of the universal covering space.

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[4] This is a lot like other categorical proofs, and basically says that simply connected spaces are *universally repelling* objects in the category of covering spaces.

3. CLASSIFICATION OF COVERING SPACES

Now for the main classification theorem:

Theorem 3.1. *Let (X, x_0) be a (pointed) space and fix $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ any universal covering space for X. Then for any $H \subseteq \pi_1(X, x_0)$, there exists a covering space $p' : (E, e_0) \to (X, x_0)$, unique up to covering isomorphism (or "equivalence" in Greenberg), such that $H = p'_*(\pi_1(E, e_0))$.*

Remark 3.2. We have noted that the group of covering transformations G of $\tilde{X}$ over X is naturally isomorphic to $\pi_1(X, x_0)$ (since our spaces are connected and locally path-connected, and $\tilde{X}$ is simply connected). It is useful to recall how this isomorphism (denote it Φ) is given: note that the basepoint $\tilde{x}_0 \in p^{-1}(x_0)$. Now, given a path class in $\pi_1(X, x_0)$, we apply the lifting lemma three times: first lift the given path class up to a path class in $\tilde{X}$ starting at point $\tilde{x}_0$, and then consider the (well-defined) endpoint $\tilde{x}'_0$ of this class. $\tilde{x}'_0 \in p^{-1}(x_0)$ as well, so we use the lifting lemma (using the fact that $\tilde{X}$ is simply connected) to lift p itself to the unique map φ such that $\varphi(\tilde{x}_0) = \tilde{x}'_0$; since $\tilde{X}$ is a universal covering we get our result using unique lifting. More details available in Oded's excellent review of covering transformations (Review #16).

Proof of the Theorem. This is long and arduous so we break it up into several steps.

STEP 1. First we construct our space E as the quotient of the universal covering space $\tilde{X}$ by an action of a group H' (the space of fibers). Let G be the group of covering transformations of $\tilde{X}$ over X and Φ be the correspondence noted in the above remark. We take H' as the subgroup of G corresponding to H i.e. $H' = \Phi(H)$. This is a group of homeomorphisms acting on $\tilde{X}$. Now consider the space $E = \tilde{X}/H' = \{H'\tilde{x} : \tilde{x} \in \tilde{X}\}$ the "orbit space" ($H'\tilde{x} = \{\psi(\tilde{x}) : \psi \in H'\}$) and give it the quotient topology with the map $\pi : \tilde{X} \to E$ by $\tilde{x} \mapsto H'\tilde{x}$ (this is well-defined since it comes from the equivalence relation on orbits). We take $p' : E \to X$ by $H'\tilde{x} \mapsto p(\tilde{x})$; this is well-defined because any element (representative) of $H'\tilde{x}$ is of the form $\psi(\tilde{x})$ for $\psi \in H'$, and by the definition of H' as a subgroup of the covering transformations, $p(\psi(\tilde{x})) = p(\tilde{x})$. By definition, then, $p' \circ \pi = p$. Since $p = p' \circ \pi$ is continuous, this shows p' is continuous by the properties of the quotient topology.

STEP 2. We must now show that p' is a covering space map. Since $p = p' \circ \pi$ and p is onto, so is p' (this doesn't give us the trouble in proving covering homomorphisms are onto, because p' is now the left map in the composition which is always onto when the composition is onto). We must show every $x \in X$ is contained in a neighborhood evenly covered by p'. Now given $x \in X$, let V be a path-connected neighborhood (such exists as we are assuming our X is locally path-connected), evenly covered by p. Choose some point $\tilde{x} \in p^{-1}(x)$ and let U be the component of $p^{-1}(V)$ containing $\tilde{x}$. Since G is the collection of all covering transformations of the universal covering space of X, G acts transitively on the fiber $p^{-1}(x)$ and moreover, since $p \circ \varphi = p$ for any $\varphi \in G$ and p restricts to a homeomorphism on U with V, φ maps U homeomorphically to the component containing $\varphi(x)$.

Since $p^{-1}(V)$ is, by hypothesis, the disjoint union of its components, if φ is not the identity, $\varphi(U) \cap U = \varnothing$ (i.e. G acts *properly discontinuously*[5] on $\tilde{X}$). Also since φ is uniquely determined by its action on a fiber by the lifting lemmas we therefore have that $p^{-1}(V) = \bigcup_{\varphi \in G} \varphi(U)$ is a precise representation of $p^{-1}(V)$ as a disjoint union. In effect this says that H' permutes the sheets of $p^{-1}(V)$. Now $p^{-1}(V) = (p' \circ \pi)^{-1}(V) = \pi^{-1}(p'^{-1}(V))$ so therefore $p'^{-1}(V) = \pi(p^{-1}(V))$ (since π is onto, $\pi(\pi^{-1}(S)) = S$ for any subset $S \subseteq E$). Our goal is

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[5] How did such a rogue term manage to find its way into topology?!?

to now represent $p'^{-1}(V)$ as a disjoint union of components each of which maps homeomorphically with V. Consider the set (not necessarily a group) of *right* cosets G/H' as an index set. Define, for each $\varphi \in G$ the set $V_{H'\varphi} = \pi(\varphi(U))$. We show that $V_{H'\varphi}$ is independent of the representative of G/H' used: if $H'\varphi = H'\psi$ i.e. $\psi \circ \varphi^{-1} \in H'$, then $V_{H'\varphi} = V_{H'\psi}$. So suppose $\psi \circ \varphi^{-1} \in H'$. Note that since H' is a group, $H'\tilde{y} = \{h(\tilde{y}) : h \in H'\} = \{h(h'(\tilde{y})) : h \in H'\} = H'h'(\tilde{y})$ for any $h' \in H'$; h' just permutes the elements of the set. Hence we have

$$\begin{aligned} V_{H'\varphi} = \pi(\varphi(U)) &= \{H'\varphi(\tilde{y}) : \tilde{y} \in U\} \\ &= \{H'(\psi \circ \varphi^{-1})(\varphi(\tilde{y})) : \tilde{y} \in U\} \\ &= \{H'\psi(\tilde{y}) : \tilde{y} \in U\} \\ &= \pi(\psi(U)) = V_{H'\psi}. \end{aligned}$$

since $\psi\varphi^{-1} \in H'$. Now we show that the $V_{H'\varphi}$ are a disjoint collection: if $e \in V_{H'\varphi} \cap V_{H'\psi}$ then $e = H'\varphi(\tilde{y})$ for some $\tilde{y} \in U$ and $e = H'\psi(\tilde{z})$ for some $\tilde{z} \in U$. By definition of the orbit equivalence relation this means $\psi(\tilde{z}) = h(\varphi(\tilde{y}))$ for some $h \in H'$ or $\tilde{z} = \psi^{-1}(h(\varphi(\tilde{y}))$. Since both $\tilde{z}, \tilde{y} \in U$ this says that $U \cap \psi^{-1}(h(\varphi(U)) \neq \varnothing$ so by proper discontinuity, $\psi^{-1} \circ h \circ \varphi = Id$. Rearranging, this says $\psi \circ \varphi^{-1} = h \in H$ so therefore $H'\psi = H'\varphi$, and hence $V_{H'\psi} = V_{H'\varphi}$. Finally, since $p'^{-1}(V) = \pi(p^{-1}(V)) = \pi\left(\bigcup_{\varphi \in G} \varphi(U)\right) = \bigcup_{\varphi \in G} \pi(\varphi(U))$ and $\pi(\varphi(U)) = V_{H'\varphi}$, and the group quotient map $G \to G/H'$ by $\varphi \mapsto H'\varphi$ is onto, we have $p'^{-1}(V) = \bigcup_{\tilde{\varphi} \in G/H'} V_{\tilde{\varphi}}$, a disjoint union. Now restricting p' to $V_{H'\varphi}$ we have that $p'(\pi(\tilde{y})) = p(\tilde{y}) \in V$ so that p' maps into V; we define $q' = \pi \circ (p|_{\varphi(U)})^{-1}$ which is continuous; we check that it is a local inverse for p' on $V_{\tilde{\varphi}}$. Given $e \in V_{\tilde{\varphi}}$ it can be represented as $\pi(\tilde{y})$ for some $\tilde{y} \in \varphi(U)$; so $q'(p'(e)) = q'(p(\tilde{y})) = \pi(p|^{-1}_{\varphi(U)}(p(\tilde{y})) = \pi(\tilde{y}) = e$ since $\tilde{y} \in \varphi(U)$ and we used the local inverse of p there. On the other hand for $y \in V$, we have $p'(q'(y)) = p'(\pi(p|^{-1}_{\varphi(U)}(y))) = p(p|^{-1}_{\varphi(U)}(y)) = y$ as $p = p' \circ \pi$.

Since p' is a covering map, it is a local homeomorphism, so the local connectivity of X implies the local connectivity of E as well. $\tilde{X}$ is simply connected, E is connected, and so by the uniqueness theorem just proved, π is a covering homomorphism. In particular, $\tilde{X}$ is also the universal cover of E and $\pi_1(E, e_0)$ is isomorphic to H'.

STEP 3. Now we show that $H = p'_*(\pi_1(E, e_0))$. So far, we have the following diagram:

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_0) \arrow[dd, "p"'] \arrow[dr, "\pi"] & \\
& (E, e_0) \arrow[dl, "p'"] \\
(X, x_0) &
\end{tikzcd}$$

Let $\Psi : \pi_1(E, e_0) \to H'$ be the isomorphism as described in the remark. We show that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(E, e_0) \arrow[r, "\Psi"] \arrow[d, "p'_*"'] & H' \arrow[d, "\iota"] \\
\pi_1(X, x_0) \arrow[r, "\Phi"] & G
\end{tikzcd}$$

(this is actually what we mean by Ψ and Φ being "natural" isomorphisms). ι denotes the inclusion map. To see the commutativity, we use techniques similar to showing that covering homomorphisms are surjective. Let $[\zeta]$ be a path class in $\pi_1(E, e_0)$. Now ζ is a representative loop based at e_0. We push ζ forward via p' to get $\sigma = p' \circ \zeta$. This lifts up to a path $\tilde{\sigma}$ with start point $\tilde{x}_0$ in $\tilde{X}$. But $p \circ \tilde{\sigma} = p' \circ \pi \circ \tilde{\sigma}$, so $p' \circ \pi \circ \tilde{\sigma} = \sigma = p' \circ \zeta$. Since $\pi \circ \tilde{\sigma}$ starts at the point e_0 (recall $e_0 = H'\tilde{x}_0 = \pi(\tilde{x}_0)$), and ζ starts at e_0 as well, this says $\pi \circ \tilde{\sigma}$ and ζ are two lifts of σ starting at the same point in E so must be equal by unique path lifting. So $\pi \circ \tilde{\sigma} = \zeta$ which in turn says that $\tilde{\sigma}$ is a lift in $\tilde{X}$ of ζ. Passing to homotopy classes, since lifts and push-forwards (mapping by p_* and p'_*) preserve homotopy classes, we have that $[\pi \circ \tilde{\sigma}] = [\zeta]$. What does this give us? $\tilde{\sigma}$ is both a lift of ζ in E and σ in X; we know that the images $\Psi([\zeta])$ and $\Phi([\sigma])$ are uniquely determined by $\tilde{x}_0$ and the endpoint $\tilde{\sigma}(1)$ (by the remark above). So both $\Phi([\sigma]) = \Phi(p'_*([\zeta]))$ and $\Psi([\zeta])$ are the same deck transformation of $\tilde{X}$ (which actually lies in H' since Ψ maps into H').

Then, since Φ is an isomorphism, $p'_* = \Phi^{-1} \circ \iota \circ \Psi = \Phi^{-1} \circ \Psi$, since ι is the inclusion. But $p'_*(\pi_1(E, e_0)) = \Phi^{-1}(\Psi(\pi_1(E, e_0))) = \Phi^{-1}(H') = H$ because Ψ is onto and $\Phi(H) = H'$ by hypothesis.

□

We now start classifying covering spaces of many things we know.

Example 3.3 (Covering spaces of the circle)**.** We calculated the universal covering space of S^1 to be $\mathbb{R}$. We note that the action of $n\mathbb{Z}$ on $\mathbb{R}$ is just translation by n, so each n yields $\mathbb{R}/n\mathbb{Z}$, which is a circle; each of these covers S^1 by the n-fold wrapping of a circle upon itself.

Example 3.4 (Covering spaces of $\mathbb{P}^n$)**.** This one is easy, since S^n is a 2-fold cover of $\mathbb{P}^n$ and so since S^n is simply connected for $n > 1$, S^n is the universal covering space, and $\pi_1(\mathbb{P}^n) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ by the antipodal map. Since $\mathbb{Z}_2$ has no other subgroups, we're done: the only two connected covering spaces of $\mathbb{P}^n$ are S^n with the antipodal map and $\mathbb{P}^n$ itself. In the case $n = 1$, $\mathbb{P}^1 = S^1$.

Example 3.5. Using nearly the same arguments as above, we can classify covering spaces of the lens spaces $L(p, q)$ for p prime and q relatively prime to p. From the construction of lens spaces as a quotient of S^3 via the group of cyclic group of homeomorphisms generated by $\epsilon : (z, w) \mapsto (e^{2\pi i/p}z, e^{2\pi i q/p}w)$ (covered in Nate's excellent Review #8), this is exactly the quotient of S^3 by the group actions described above (we need to use the fact that the group action is properly discontinuous, since we are not starting at the outset with S^3 already being a covering space of something). By assuming proper discontinuity, by a similar method of proof above, we can show that the quotient map is indeed a covering map, so S^3 is a covering space for $L(p, q)$. Since S^3 is simply connected (Evan's excellent Review #15), S^3 is therefore the universal covering space and $G = \langle \epsilon \rangle$ is isomorphic to the fundamental group of $L(p, q)$. Since p is prime, and q is relatively prime to p, the size of G is p so that $G \cong \mathbb{Z}_p$, a cyclic group of prime order[6] and hence it has no subgroups other than the trivial one. So S^3 with the covering map and $L(p, q)$ with the identity map are the only (covering isomorphism classes of) covering spaces for $L(p, q)$.

Example 3.6 (Covering spaces of the torus $\mathbb{T} = S^1 \times S^1$)**.** Now for something not nearly as trivial as our above examples. It is therefore completely appropriate that we use our favorite nontrivial example: the torus. In this case the universal covering space is $\mathbb{R}^2$ via the standard exponential maps $(x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y})$. Abbreviate $t \mapsto e^{2\pi i t}$ as $\varepsilon(t)$ for short. Since products of covering spaces are

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[6] Following the non-blasphemous convention that $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. As an interesting side note, I've seen ${}_p\mathbb{Z}$ used to denote the subgroup annihilated by multiplication by p. Cute. Very cute.

covering spaces, we can take a guess at what the covering spaces of $\mathbb{T}$ might be: the entire plane, an infinite cylinder (corresponding to one factor having the universal cover and the other having the circular cover), or the torus itself (corresponding to both factors having circular covers).

This gives us some idea of the flavor of what is going on. But we can be more precise; we find that the covering maps are a little more exotic than we might suspect at first. The action of subgroups of $\mathbb{Z}\times\mathbb{Z}$ corresponds to translation by vectors with integral coordinates in $\mathbb{R}^2$. Since $\mathbb{Z}\times\mathbb{Z}=\mathbb{Z}\oplus\mathbb{Z}$, the free abelian group on 2 generators, any subgroup must have 0, 1, or 2 generators.

CASE 1. This gives the trivial subgroup, which always corresponds to the universal cover.

CASE 2. We treat the case of two generators; the one-generator case will be a simple adaptation of the method used here). Given two generators (a,b) and (c,d) (which are by hypothesis linearly independent), we have the *inverse* change-of-basis matrix $T^{-1}=\begin{pmatrix} a & c \\ b & d\end{pmatrix}$. For topological purposes, T is best viewed as a homeomorphism of $\mathbb{R}^2$ with itself rather than trying to clumsily represent it as the same $\mathbb{R}^2$ algebraically as $\mathbb{R}\langle a,b\rangle\oplus\mathbb{R}\langle c,d\rangle$. Let E be the orbit space on $\mathbb{R}^2$ modulo the action of $\mathbb{Z}\langle a,b\rangle\oplus\mathbb{Z}\langle c,d\rangle$. Note that $p\circ T:\mathbb{R}^2\to\mathbb{T}$ is a covering map since T is a homeomorphism and p is a covering map. We aim to define Φ to give us the following commutative diagram:

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[dd, "p"'] \arrow[dr, "\pi"'] \arrow[drr, "p\circ T"] & & \\
& E \arrow[dl, "p'"'] & \mathbb{T} \arrow[l, dashed, "\Phi"'] \arrow[dll, "\tilde{p}'"] \\
\mathbb{T} & &
\end{tikzcd}$$

that is, we want to "represent" the strange orbit space E with something more palatable, the torus itself (via the map Φ), with a different, but isomorphic covering map (call it $\tilde{p}'$). We define Φ the obvious way by "brute force": given $(z,w)\in\mathbb{T}$ we take representatives (s,t) such that $(e^{2\pi i s},e^{2\pi i t})=(z,w)$ (i.e. choose (s,t) in the fiber $p^{-1}(z,w)$. We now apply the inverse change-of-basis matrix T^{-1} to yield $(as+ct,bs+dt)$ and take the equivalence class of this modulo the subgroup generated by (a,b) and (c,d). This is independent of the choice of representative, since if (s',t') is any other pair such that $p(s',t')=(z,w)$, we have $(s-s',t-t')\in\mathbb{Z}^2$. Since T^{-1} is linear this says $T^{-1}(s-s',t-t')\in T^{-1}(\mathbb{Z}^2)$. But $T^{-1}(\mathbb{Z}^2)=\{(x,y):T(x,y)\in\mathbb{Z}^2\}$ which is the set of all points, when the basis is changed, ends up on the lattice. This is precisely $\mathbb{Z}\langle a,b\rangle\oplus\mathbb{Z}\langle c,d\rangle$ by definition of changing bases. So therefore when taking π the $(s-s',t-t')$ disappears and the map Φ is well-defined. By construction, then, given $(x,y)\in\mathbb{R}^2$, $\Phi(p(T(x,y)))=\pi(x',y')=\pi(x,y)$ because x' and y' differ from x and y by some element of $T^{-1}(\mathbb{Z}^2)$.

Now $p\circ T$ gives $\mathbb{T}$ the quotient topology since T is a homeomorphism and p is a quotient map. So by properties of quotient maps, Φ is continuous because $\pi=\Phi\circ(p\circ T)$ is. The inverse of Φ is given by taking representatives in the fibers of π; it is well defined because T sends $\mathbb{Z}\langle a,b\rangle\oplus\mathbb{Z}\langle c,d\rangle$ to $\mathbb{Z}^2$ which is the kernel of p. Since E is Hausdorff and $\mathbb{T}$ is compact, Φ is a continuous bijection hence a homeomorphism. Now we define $\tilde{p}':\mathbb{T}\to\mathbb{T}$ by $\tilde{p}'=p'\circ\Phi$. This is a covering map because p' is and Φ is a homeomorphism. The diagram commutes because $\tilde{p}'\circ p\circ T=(p'\circ\Phi)\circ(\Phi^{-1}\circ\pi)=p'\circ\pi=p$. This allows us to (finally!) compute $\tilde{p}'$ explicitly: given $(z,w)\in\mathbb{T}$ we have that it comes from, as before, $(s,t)\in\mathbb{R}^2$ via p, and then applying T^{-1} we have that it is $(as+ct,bs+dt)$. Taking p, we have that $\tilde{p}'(z,w) = (\varepsilon(as+ct), \varepsilon(bs+dt)) = (\varepsilon(as)\varepsilon(ct), \varepsilon(bs)\varepsilon(dt)) = (z^a w^c, z^b w^d)$ because $(z,w) = (\varepsilon(s), \varepsilon(t))$ by definition. Therefore the covering space corresponding to a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by two elements (a,b) and (c,d) yields the covering space $\tilde{p}' : \mathbb{T} \to \mathbb{T}$ by $p(z,w) = (z^a w^c, z^b w^d)$.[7]

CASE 3. For one generator, the situtation is a little simpler. Given the generator (a,b) (nonzero by hypothesis), we take another basis vector $(-b,a)$. This is just rotating (a,b) by 90 degrees counterclockwise—the motivation is that since the orbits are translations given by the single vector (a,b), we can think of the orbit space as a cylinder with an axis in the direction of $(-b,a)$, as all the lines of $\mathbb{R}^2$ in this direction are identified with their parallel translates by $n(a,b)$ for $n \in \mathbb{Z}$. Therefore we change the basis to $\{(a,b),(-b,a)\}$ (in that order), which is guaranteed to be linear independent by perpendicularity. Let T be the change of basis transformation as before and F be the orbit space. We need a similar diagram; however, instead of using $p \circ T$, taking $q = \varepsilon \times Id$ i.e. the map given by $(x,y) \mapsto (\varepsilon(x), y)$ we now use $q \circ T$ (which is also a quotient map):

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[dr, "\pi"] \arrow[drr, "q\circ T"] \arrow[dd, "p"'] & & \\
& F \arrow[dl, "p'"] & S^1 \times \mathbb{R} \arrow[l, dashed, "\Psi"'] \arrow[dll, "q'"] \\
\mathbb{T} & &
\end{tikzcd}$$

We define Ψ in a similar manner: given (z,t) first get a fiber via q, which will be (s,t) where $\varepsilon(s) = z$ and then apply T^{-1} and finally taking π to get $\pi(as-bt, bs+at)$; this is well-defined because now we're only dealing with the subgroup $\mathbb{Z}\langle a,b\rangle$, and the ambiguity of representation lies in s only. Since the difference of s with another representative s' is an integer, which gets mapped to a multiple of (a,b) this difference is of course inside $\mathbb{Z}\langle a,b\rangle$. So Ψ is well defined and $\pi = \Psi \circ q \circ T$. Similar arguments show that Ψ is in fact continuous ($q \circ T$ being a quotient map and its composition being π). We similarly explicitly construct the inverse of Ψ by first taking a fiber of π, pushing over with T, and then with q. Then this is well defined by the same arguments, and since π is a quotient map, this inverse is also continuous. Defining $q' = p' \circ \Psi$ gives the commutativity of the diagram, and so we compute $q'(z,t) = p(as-bt, bs+at) = (\varepsilon(as)\varepsilon(-bt), \varepsilon(bs)\varepsilon(at)) = (z^a \varepsilon(t)^{-b}, z^b \varepsilon(t)^a)$. So $S^1 \times \mathbb{R}$ with the covering map $q'(z,t) = (z^a e^{-2\pi i b t}, z^b e^{2\pi i a t})$ works.[8]

Example 3.7 (Covering Spaces of $S^1 \vee S^1$)**.** If you thought the subgroup structure of $\mathbb{Z} \oplus \mathbb{Z}$ was bad, $\mathbb{Z} * \mathbb{Z}$ is much worse! The universal cover of this space is the Evil Christmas Tree of Death (the ECToD, shown right before the introduction); see Hatcher §1.3. for details.

I have found a marvelously cute description of all covering spaces of $S^1 \vee S^1$ but I'm afraid this margin is just too small to contain it...

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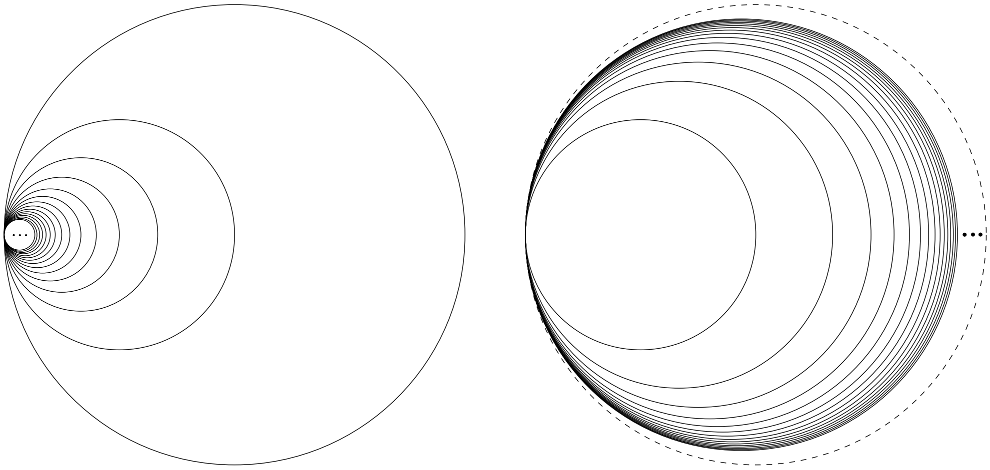
[7] In particular if $b = c = 0$ then this gives us $(z,w) \mapsto (z^a, w^d)$ which is the torus wrapping around itself a times in the first factor and d times in the second, corresponding to the product of two finite-sheeted covering spaces of the circle. That w and z get mixed together in the general case is just a reflection of non-perpendicularity.

[8] In particular if $b = 0$ then $(z^a, e^{2\pi i a t})$ is a cover corresponding to $\mathbb{R}$ in the second factor and a circular cover in the first.



(a) Hawaiian Earring

(b) A bounded subset of $\mathbb{R}^2$ homotopy-equivalent to $\bigvee_{n=1}^{\infty} S^1$

FIGURE 1. Comparison of two infinite wedges of circles.

APPENDIX A. BRIEF DESCRIPTION OF THE CONSTRUCTION OF THE UNIVERSAL COVERING SPACE

The idea of the construction of the universal covering space is simple, although some of the details in the construction can look complicated. We give a brief overview to get a feeling for it but skip over the details.

Definition A.1. B is *semilocally simply connected* if B admits a basis of open sets such that any loop in a basis element U can be shrunk to a point by a homotopy *in* B (but not necessarily in U; if this is the case, i.e. each U is simply connected, then we just say B is *locally simply connected*). In more algebraic terms, semilocally simply connected means that, at b, the inclusion-induced map $\pi_1(U,b) \xrightarrow{\iota_*} \pi_1(B,b)$ is the zero map, while locally simply connected means that $\pi_1(U,b) = 0$ for all basis elements U.

Before continuing we'll give a few cute examples to familiarize ourselves with semilocally simple connectivity (it's quite a mouthful).

Example A.2. Every (locally path-connected) simply connected space X is semilocally simply connected, since a loop in any basis element of the space can be deformed to a point in X (or, any inclusion-induced map into $\pi_1(X,x)$ must be trivial).

Example A.3. An example of a space that is *not* semilocally simply connected is the incredible countable shrinking wedge of circles (sometimes called the "infinite Hawaiian earring") which consists of the circles of radius $1/n$, $n \geq 1$ all tangent at the origin considered as a subspace of $\mathbb{R}^2$ (Figure 1(a)). Any neighborhood of the origin contains all but finitely many of the circles, so a loop going around one of those circles, for example, cannot be shrunk to a point in any part of that space. This looks like the countable wedge sum of circles, but it is not, since a neighborhood of the join point in that case is contractible (since the circles are not simultaneously shrinking to zero size). For comparison, a bounded subspace of $\mathbb{R}^2$ homotopy-equivalent to a countable wedge sum of circles $\bigvee_{n=1}^{\infty} S^1$ is given in Figure 1(b) (the circles approach the outermost dashed circle but the dashed circle is *not* included; if we did, this gives an example of a space that isn't locally path connected).[9]

Now we consider the cone CX of this space. CX is contractible, hence simply connected, so in turn semilocally simply connected. The "origin" in X gets mapped to a line segment to which all the shrinking cones are tangent. Any neighborhood of a point on this line which excludes the "vertex" is not simply connected, although a loop may be deformed to a point by going outside the neighborhood, through the common vertex of the cone, and back along the outermost cone. Hence CX is semilocally but not locally simply connected.

Now we describe the process of constructing the universal covering space. It is actually kind of cool. Again this is only an overview so details will be glossed over. Let X be the space and choose a base point $x_0 \in X$. Then any $x \in X$ has a path $\sigma_{x_0,x}$ starting at x_0 and ending at x. $\sigma_{x_0,x}$ of course has a path homotopy class $[\sigma_{x_0,x}]$, which are determined by the endpoints x_0 and x. Consider the collection $\tilde{X} = \{[\sigma_{x_0,x}] : x \in X\}$, i.e. elements of this set (which will be the *points* in what is to become our universal covering space) are path classes. There is a nice map $p : \tilde{X} \to X$ given by sending each homotopy class to its endpoint (again, well-defined because path homotopies preserve endpoints). This will be our covering map. We choose our basepoint $\tilde{x}_0$ to be the path class of the constant path x_0.

We define a topology on $\tilde{X}$ with the following motivation: the "number of sheets" of the cover at x (cardinality of the fiber $p^{-1}(x)$) should be just the number of different homotopy classes connecting x_0 to x, and the notion of "nearness" in $\tilde{X}$ should include the homotopy classes of paths ending at points near to x, but yet also forces different homotopy classes ending at x to be "far away" that is, far enough so that neighborhoods of distinct path classes at x can be disjoint. This is where the local path-connectivity and semilocally-simple-connectivity come into play—taking a basis of path-connected sets satisfying the semilocally simple-connectivity property, we construct a neighborhood basis at each $[\sigma_{x_0,x}]$ where each basis element is given by path classes that "continue" from x into each basis element (in X) U containing x. It's a little work to show that this satisfies the properties of a neighborhood basis. It works out very nicely because points "near" to $[\sigma_{x_0,x}]$ are included any of the little extra tails introduced by a path in U doesn't jump wildly to another homotopy class since two paths in U with the same endpoints will still be homotopic in X—by its definition U can't contain more than one homotopy class. In particular this makes the projection map continuous because the whole neighborhood of paths defined by U will map into U. And similarly other homotopy classes from x_0 to x attached to the same neighborhood U will comprise the disjoint union that $p^{-1}(U)$ is to be. If p is just restricted to one neighborhood, endpoints of all the nearby paths is easily verified to be in 1-1 correspondence with U since U doesn't add any additional homotopy classes.

Path-connectivity of $\tilde{X}$ is proved by taking paths in X and taking "snapshots": given $\sigma : [0,1] \to X$, we define $\Sigma : [0,1] \to \tilde{X}$ at time t to be the path class of the "snapshot" of the traversal of σ from time 0 to time t. The fact that $\tilde{X}$ is simply connected is relatively clear—by making different homotopy classes distinct points, a loop based at a point necessarily stays in the same class, hence is homotopic to the point itself.

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[9] A genuine countable wedge of circles cannot be embedded as a subspace in $\mathbb{R}^2$ as it is not first-countable. I originally thought the one on the right would work, but I've kept it in because it looks cool anyway…