Math 20E Midterm 2 Study Notes

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1 Multiple Integration: Overview of Concepts

The story so far: In the Beginning the Integral was created. This made (and still makes) a lot of people very angry and was (and still is) widely regarded as a bad move (curse you, Newton! Curse you!!!). Ok, not that story. The point of the material covered since the last midterm (Homeworks 5, 6, 7, 8) has been to extend the idea of integration over higher-dimensional domains. We'd like to integrate over domains that aren't always nice and straight, but rather, regions that have curvy boundaries and even regions with curvy interior. With the mass of different notation, techniques, etc. it is easy to get confused. You shouldn't feel too bad if you are a bit lost. Some burning questions you might have are "What the heck is a surface integral? Vector surface integral? What's a Jacobian and why do we need it, and why does this $\|\mathbf{T}_u \times \mathbf{T}_v\|$ make a sudden, random appearance?" I'll try to give a summary of all these things, as it is always helpful to try to get the big picture.

1.1 Chapter 5

The material of chapter 5 is integration over flat 2 and 3-dimensional domains, but even though they are flat on the inside, they can have funny boundaries. This issue never happens in 1 dimension because the boundary of an interval is always going to be 2 points![1] But 2-dimensional domains can have 1-dimensional boundaries, and these can wiggle around. This wiggle-around business corresponds to making the bounds of integration have complicated bounds that are *functions* of the variables that haven't already been integrated.

The regions of integration are usually described by some kind of inequalities: x lies in between such and such, then y also lies between such and such, which may also depend on what x is, etc. What the book describes as a "simple region" is that we can choose one variable independently, say x to lie in between two *constants*, and then the other variable can lie in between two numbers depending on the first variable only. Often we can actually convert from the description in one variable to the other. Like a disk in the plane (area bounded by a circle), which is usually written $\{(x, y) : x^2 + y^2 \leq 1\}$ can also be represented as the region $\{(x, y) : -1 \leq x \leq 1 \text{ and } -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$, or $\{(x, y) : -1 \leq y \leq 1 \text{ and } -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\}$. The heart of the problem of interchanging the order of integration is basically converting from one description of the domain to the other.

1.2 Chapter 6

Chapter 6 introduces a way to warp some kinds of domains with boundaries that are hard to work with, into domains that are easier to handle. For example, integration over rectangles is always nice because the bounds of integration are constants. Also, it may significantly simplify the integrand (function to be integrated) itself. This leads to the idea of "transformations"—functions taking a domain and changing it into something else. The confusing issue here is which way we're going—since transformations can go forward and back. In other words, what the heck are D and D^*. So the idea is, we assume that as given, our problem is described in the (x, y) or (x, y, z) domain and looks complicated.

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[1] Actually, in the study of Lebesgue integration—covered in the class I'm taking right now—we can integrate over crazy weird domains even in one dimension; instead of intervals, for example, we can integrate over a scattered dust of points. Thank goodness we usually don't have to actually *calculate* these things!

The idea is if we can express $(x, y) = T(u, v)$ for some transformation T, and the domain where it comes from can be expressed more simply, then we rewrite the domain D of the the given variables (x, y) in terms of the domain D^* of the new variables (u, v). So really, computing the domain is running the T backwards from (x, y) to get (u, v). More relevant to computation: if we have $D = \{(x, y) : a \leq x \leq b \text{ and } f(x) \leq y \leq g(x)\}$ (a domain that is simple with respect to x), and we have $(x, y) = T(u, v) = (T_1(u, v), T_2(u, v))$, then, to compute D^* we substitute the x and y directly with $T_1(u, v)$ and $T_2(u, v)$ directly in the expression. So $D^* = \{(u, v) : a \leq T_1(u, v) \leq b \text{ and } f(T_1(u, v)) \leq T_2(u, v) \leq g(T_1(u, v))\}$. Then this satisfies $T(D^*) = D$ or equivalently $D^* = T^{-1}(D)$ the inverse function. The integrand will also be a function in terms of (x, y) so we also change it to be in terms of (u, v) as well. Finally, the Jacobian is necessary as a "correction factor"—integration is, at its heart, a system of measurement. In particular, it's all about area and volume. So if our transformation warps space in a funny way, it will change area. The Jacobian corrects this and is analogous to the derivative of a composition of functions $(f \circ g)'(t)$ needing the extra $g'(t)$ in is chain rule expansion $f'(g(t))g'(t)$. In other words, to finally get to the point, this method of integration is the multivariable "u-substitution"—except now it's really (u, v)- (or even (u, v, w)-) substitution. It's summed up on the next formula:

$$\iint_D F(x, y)\, dx\, dy = \iint_{D^*=T^{-1}(D)} F(T(u, v)) \det[\mathbf{D}T(u, v)]\, du\, dv$$

A "real" example will follow in a moment.

The most well-known, and indeed most useful, are the coordinate transformations introduced the 2nd or 3rd week—polar coordinates (for 2-D problems) and cylindrical and spherical (for 3-D problems), which are $P(r, \theta) = (r\cos\theta, r\sin\theta)$, $C(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$, and $S(\rho, \varphi, \theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$. An illustrative example, then, is $\iint_{x^2+y^2\leq 1} e^{x^2+y^2}\, dx\, dy$. Recognizing $x^2 + y^2$, because we know this is r^2, it swells in our hearts to convert to polar coordinates. So how do we re-express the domain $D = \{(x, y) : x^2 + y^2 \leq 1\}$ in terms of r and θ? We plug in what x is in terms of it, wherever it occurs, and similarly for y: $D^* = \{(r, \theta) : r^2\cos^2\theta + r^2\sin^2\theta \leq 1\}$. Recognizing $\sin^2\theta + \cos^2\theta = 1$ we have $D^* = \{(r, \theta) : r^2 \leq 1\}$ or since r is always positive, $D^* = \{(r, \theta) : 0 \leq r \leq 1\}$. But what about θ? If no restriction is made on θ, its "natural range" is $[0, 2\pi]$. The natural range of r which you will usually not have occasion to use unless you do improper integrals, is $0 \leq r < \infty$. In most cases you will end up explicitly computing a range for r and leave θ alone. Finally, the Jacobian:

$$\mathbf{D}T(r, \theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$$

which has a determinant of $r\cos^2\theta + r\sin^2\theta = r$. So the integral is now

$$\iint_{r\leq 1} e^{r^2} r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 e^{r^2} r\, dr\, d\theta$$

Now that this beast has been reduced to two 1-dimensional integrals, you can bribe your friends in 20B to finish it off for you (except, of course, on the exam, but you won't encounter nasty integrals there, anyway). For the interested, it's a 1-variable u-substitution. For your reference the Jacobians for Cylindrical and Spherical are, respectively r and $\rho^2\sin\varphi$. These are worth memorizing.

1.3 Chapter 7

The subject of interest here is integration over domains where, not only the boundaries can be curvy, but the interiors can curve around, too. And here, things get interesting even in 1 dimension, since the interior of a 1-dimensional thing can curve around (as anyone who's ever tied shoelaces knows). One might think that totally different methods are needed here. Different, yes. Totally, no. We need a way to speak of the domain of integration. This is usually given as a parameterization (a word I only since recently have been able to pronounce without stumbling over something). A parameterization is actually just a transformation (like in Chapter 6) except the range (where you end up) of the transformation can have more dimensions than the domain (where you start from)—this expresses the fact that the flat thing is being put into a dimension higher up, which allows it to wiggle around. I will talk about actually finding parameterizations—half the battle in doing line and surface integrals—in an upcoming section, with a couple of good examples.

In integration over these surfaces, therefore, we find a transformation from the line or plane to this object that wiggles in a higher dimension. It turns out the only way you can actually compute the integral is to use the transformation (parameterizaton) to *convert the integral into one that can be done in flat space*. Then we've reduced it to Chapters 5 and 6. This is the essential relation between Chapter 7 and Chapters 5 and 6. Parameterizations really being transformations in disguise, then, raises the question: what is the Jacobian? It is the $\|\mathbf{T}_u \times \mathbf{T}_v\|$ (or in the case for curves, $\|\mathbf{c}'(t)\|$). These are the only kinds of parameterized domains we consider in this class.[2]

Also in this chapter we are introduced to two different types of integrals over curvy domains: path vs. line integrals, and scalar vs. vector surface integrals. Actually it is the path/scalar integral that is the basic thing. Integrals of vector fields over curvy domains are just one very useful application of this concept—we integrate a particular scalar function that comes from the dot product of the vector function with some other vector field. However, in the process, a computational shortcut comes up that makes life easier, but at the same time can be confusing because you end up thinking that it is yet another different thing to do. Paradoxically integrals of vector fields (which sound harder) end up being easier to compute because of this shortcut—it eliminates that square root term (and as we all know, square roots cause big trouble when it comes to integration). What happens is this: we usually integrate a vector field dot-producted with the unit tangent vector (for curves), or the unit normal vector (for surfaces), and these correspond to $\mathbf{t} = \mathbf{c}'(t)/\|\mathbf{c}'(t)\|$ and $\mathbf{n} = (\mathbf{T}_u \times \mathbf{T}_v)/\|\mathbf{T}_u \times \mathbf{T}_v\|$. Then when doing the normal surface integral, the denominator cancels the "Jacobian" term. To sum up in formulæ, if C is a curve parameterized by $\mathbf{c}(t)$, S is a surface parameterized by $\mathbf{T}(u, v)$, and $\mathbf{F}$ a vector field,

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_C \mathbf{F}\cdot\mathbf{t}\,ds = \int_a^b \mathbf{F}(\mathbf{c}(t))\cdot\frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}\|\mathbf{c}'(t)\|\,dt = \int_a^b \mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t)\,dt$$

where $[a, b]$ is the domain of $\mathbf{c}$, and

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}(\mathbf{T}(u,v))\cdot\frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|}\|\mathbf{T}_u\times\mathbf{T}_v\|\,du\,dv = \iint_D \mathbf{F}(\mathbf{T}(u,v))\cdot(\mathbf{T}_u\times\mathbf{T}_v)\,du\,dv$$

where D is the domain of $\mathbf{T}$. To summarize in less scary-looking formulæ, the heart of line and surface

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[2] If anyone ever goes and parameterizes 3-space object (u, v, w) to wiggle in 4-space, it would be

$$\left\|\det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \boldsymbol{\ell} \\ \partial T_1/\partial u & \partial T_2/\partial u & \partial T_3/\partial u & \partial T_4/\partial u \\ \partial T_1/\partial v & \partial T_2/\partial v & \partial T_3/\partial v & \partial T_4/\partial v \\ \partial T_1/\partial w & \partial T_2/\partial w & \partial T_3/\partial w & \partial T_4/\partial w \end{bmatrix}\right\|$$

It's even more fun parameterizing 2D surfaces in 4-space (this requires 6 (gasp!) components!).

integrals lies in the ds's and dS's, the central four formulæ are:

$$ds = \|\mathbf{c}'(t)\|\, dt \tag{1.1}$$

$$d\mathbf{s} = \mathbf{t}\, ds = \mathbf{c}'(t)\, dt \tag{1.2}$$

$$dS = \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv \tag{1.3}$$

$$d\mathbf{S} = \mathbf{n}\, dS = (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv \tag{1.4}$$

1.4 Other random conceptual notes

You will probably encounter problems asking about area of a region in the plane or area of a surface in space (and similarly, volume of a region in space, and arc length for curves). Integrals, as mentioned before, are about measurement—lengths, areas, volumes (the class in analysis I am taking is concerned with a lot of *measure theory* which is basically an abstract, super-fortified theory of integration). The most basic example of something you could compute with integrals, then, are areas. And indeed, in the beginning way back when, you learned about the integral as "the area under the curve." Going one dimension up, we can integrate the volume under a surface. But this is only a special case of the integral as measurement in one dimension up. If you integrate the constant function 1 (i.e. no integrand at all), then it will give you the measure of the domain of integration—whether it is flat, has curvy boundaries, or has curvy interior. Indeed the "area under the curve" interpretation of a one-variable integral $\int_a^b f(x)\, dx$ is really the area of the region in the plane defined by $a \le x \le b$, and $0 \le y \le f(x)$. so that we can simply recast the integral

$$\int_a^b f(x)\, dx = \int_a^b \int_0^{f(x)} 1\, dy\, dx$$

(to make a simple thing more complicated =). Also $\int_a^b 1\, dx = b - a$ as I hope you can compute, which is the length of $[a, b]$. Similarly, area of a region is the double integral of 1 over that region, surface area of a parameterized surface is the surface integral of 1, length of a curve is the path integral of 1, and volume of a region in space is the triple integral of 1. Now when we have an integrand which is some function G, this is really assigning a kind of "weighting function" which somehow makes the area in one part of the domain "count more" than the other. And if G becomes negative some parts count as negative and actually subtract from the total contribution. This is why, for example, we can calculate things like masses from "density functions"—because this says, really, parts of your domain are denser than others and can contribute more to the total mass of the region you're calculating over. This is the big, conceptual thing about integration, and it is the reason why the Integral was invented (despite the fact that it still makes a lot of people angry).

2 Examples and Applications

2.1 Finding Transformations and Parameterizations

Finding transformations and parameterizations can be a tough thing to do and it is half the battle in computing these integrals. There are no general rules of thumb for changing an arbitrary thing into another arbitrary thing—there are just too many different possibilities. We have to do things like take advantage of symmetry, linearity, or whatever simplifying assumption we can get our hands on. Generally for transformation problems (same dimension to same dimension) you are given some transformation and asked to compute the image of some domain, or the Jacobian, etc. The most useful “find a transformation” kind of problem is using polar, cylindrical, and spherical coordinates. Generally you do this when you see expressions such as x^2+y^2 or $x^2+y^2+z^2$ in either the integrand or the description of the domain, or some other thing that makes it obvious that these transformations are required. The other thing you will often see is scaling of the coordinates. Here is a not-completely-trivial example given in the homework:

Example 2.1.1. Compute the volume of the ellipsoid

$$E = \{(x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$$

and also calculate $\iiint_E \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dV$.

This almost looks like a $x^2 + y^2 + z^2$ expression, but not quite. But it is still the sum of 3 squares, except some things are scaled. This suggests the substitution $u = x/a$, $v = y/b$ and $w = z/c$. Then indeed this reduces $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ to $u^2 + v^2 + w^2$. Our new domain $E^* = \{(u,v,w) : u^2 + v^2 + w^2 \le 1\}$, obtained by simply substituting u for x/a, v for y/b, and $w = z/c$. in the expression defining E. So our domain is now a sphere. We must compute the Jacobian, except now it has to be in terms of x, y, z, so we have $x = au$, $y = bv$, and $z = cw$. The matrix $\partial(x,y,z)/\partial(u,v,w)$ is therefore matrix given by a, b, and c on the diagonal. So the determinant is abc. For the volume problem we have

$$V(E) = \iiint_E 1\, dV = \iiint_{E^*} abc\, dV = abc \iiint_{E^*} 1\, dV$$

which is abc times the volume of the unit ball E^* which we already know to be $\frac{4}{3}\pi$. So the answer is $\frac{4}{3}\pi abc$. On the other hand, to do the other guy we have

$$\iiint_E \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dV = \iiint_{E^*} (u^2 + v^2 + w^2) abc\, dV = abc \iiint_{E^*} (u^2 + v^2 + w^2)\, dV$$

Now this as it stands is hard to evaluate. But what swells in our heart when we see the sum of 3 variables squared? Spherical coordinates. Except this time we have to use $u = \rho \sin\varphi \cos\theta$, $v = \rho \sin\varphi \sin\theta$ and $w = \rho\cos\varphi$, instead of $x =$, $y =$, and $z =$ the above (the spherical coordinate *transformation*) is useful for variables by any name. Hence

$$abc \iiint_{E^*} (u^2 + v^2 + w^2)\, dV = abc \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2(\rho^2 \sin\varphi)\, d\rho\, d\varphi\, d\theta$$

(recall that the $\rho^2 \sin\varphi$ is the Jacobian, and the unit ball has the description $\{(\rho,\varphi,\theta) : 0 \le \rho \le 1, 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}$). We can separate these integrals out to get $2\pi\left(\int_0^\pi \sin\varphi\, d\varphi\right)\left(\int_0^1 \rho^4\, d\rho\right)$. $\int_0^\pi \sin\varphi = -\cos(\pi) + \cos(0) = 2$ and finally the last one is $1^5/5 = 1/5$. The total answer is then $\frac{4}{5}\pi\, abc$ (congratulations, you’ve just computed the rotational inertia of an ellipsoid with respect to the origin in 4-space). The astute reader may have noticed that we could have killed two birds with one stone via the transformation $x = a\rho \sin\varphi \cos\theta$, $y = b\rho \sin\varphi \sin\theta$ and $z = c\rho\cos\varphi$ which gets rid of the intermediate u, v, and w, but I do think it is better, when seeing this kind of thing for the first time, to look at it in two steps as each one is conceptually easier. Plus it gives us the opportunity to take a look at two different transformations—giving us practice.

For parameterizations, an important example that can always be done, if we're given some simple equation defining our surfaces, is the GRAPH PARAMETERIZATION. Given something that allows us to solve for z, say $z = f(x, y)$, an automatic parameterization that will work is $(x, y, f(x, y))$ (or, $(u, v, f(u, v))$ if you prefer to keep letters in the parameter domain vs. letters in the range separate). Note one caveat is that it may not parameterize the whole surface—this happens when you "lose information" when solving for z—for example, if you have z^2 = something then taking the square root throws the negative solution away. The only recourse is to say that the surface is "piecewise parameterized" with two different parameterizations corresponding to two different ways of solving for z. The effect this has on integration is simply that you have to evaluate the integrals separately for each piece and add the results together.

Also, if you can solve for, say x instead of z, so that you have $x = g(y, z)$, then the graph parameterization is $(g(y, z), y, z)$ or $(g(u, v), u, v)$. This is because the x goes in the first slot. Similarly if $y = h(x, z)$ then the parameterization is $(x, h(x, z), z)$. This graph parameterization, though in some sense "trivial" is nevertheless very important (and useful!).

Now there are some surfaces that have nice descriptions in cylindrical and spherical coordinates, say that our surface is the "spherical graph" $\rho = f(\varphi, \theta)$. Now we'd like to say something like "Oh, well, it's a graph! We can write it $(f(\varphi, \theta), \varphi, \theta)$." Now that's *almost* true, but really it is a graph in *spherical coordinates*. Recall that a parameterization of a surface is always given as x = something, y = something, and z = something (because the "vector" defining the parameterization is the radius vector from the origin to the point on the surface). Therefore we have to then transform our graph $(f(\varphi, \theta), \varphi, \theta)$ back into rectangular coordinates using formula we all know and hopefully love, $S(\rho, \varphi, \theta) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\theta)$. In effect this is taking a graph parameterization and composing it with the spherical coordinates transformation—we substitute $f(\varphi, \theta)$ wherever we find ρ in the transformation. So therefore the total parameterization for $\rho = f(\varphi, \theta)$ is

$$(f(\varphi, \theta) \sin\varphi \cos\theta,\ f(\varphi, \theta) \sin\varphi \sin\theta,\ f(\varphi, \theta) \cos\varphi).$$

Similarly if $\varphi = g(\rho, \theta)$ we have

$$(\rho \sin(g(\rho, \theta)) \cos\theta, \rho \sin(g(\rho, \theta)) \sin\theta, \rho \cos(g(\rho, \theta))).$$

Finally, cylindrical coordinates. This actually is the coordinate system of choice for surfaces of revolution. Indeed the formula given in the book the parameterization of a surface of revolution is simply composing with the cylindrical coordinate transformation (one caveat: the book's formula defines it for revolution about the x-axis—therefore the formula they give is cylindrical coordinates with everything on its side, via the transformation $(x, r\cos\theta, r\sin\theta)$ instead of the usual formula). If we have $z = f(r, \theta)$ then composing the graph parameterization $(r, \theta, f(r, \theta)$ with the usual spherical coordinate transformation, we have the parameterization

$$(r\cos\theta, r\sin\theta,\ f(r, \theta))$$

And for $r = g(\theta, z)$,

$$(g(\theta, z)\cos\theta, g(\theta, z)\sin\theta, z)$$

Note that since $r \geq 0$ the bonus for this parameterization is that, when solving for r, you can take square roots with impunity and you don't need to do any of this piecewise business. Finally I will leave it as an exercise for the reader if we have $\theta = h(z, r)$ for some h. The same applies for curves in the plane except it's simpler because there aren't so many damn things to keep track of. Enough theory. Examples.

Example 2.1.2 (Archimedian Spiral). Everyone's favorite spiral in the plane is the spiral given by $r = \theta$. That is, r, the planar distance from the origin, gets larger in direct proportion θ. So we have the "graph parameterization" (θ, θ) (or (r, r), it does not matter which). Changing to the actual 2D parameterization we see that $(\theta \cos\theta, \theta \sin\theta)$ is it.

Example 2.1.3 (Sphere). Ahh, this is the prime example and it will get you through many, many homework problems. It is possible to do parameterizations in all 3 different ways here. Suppose our sphere is given by $x^2 + y^2 + z^2 = R^2$ where R is the radius. We solve for z, so that $z = \sqrt{R^2 - x^2 - y^2}$ and $z = -\sqrt{R^2 - x^2 - y^2}$ parameterize the upper and lower hemispheres, respectively. So the whole sphere is given by two graph parameterizations

$$T_1(x, y) = \left(x, y, \sqrt{R^2 - x^2 - y^2}\right)$$
$$T_2(x, y) = \left(x, y, -\sqrt{R^2 - x^2 - y^2}\right)$$

over the domain $x^2+y^2 \leq R^2$ (note that the domain of a graph parameterization is always the projection of the thing into the xy-plane, which is effectively accomplished by setting $z = 0$). For cylindrical coordinates we have $z^2 + r^2 = R^2$. If we solve for z again, we will get exactly the same as above except with r^2 replacing $x^2 + y^2$. So our "graph parameterization" is $(r, \theta, \sqrt{R^2 - r^2})$ and the so that we have

$$T_1(r, \theta) = \left(r \cos\theta, r \sin\theta, \sqrt{R^2 - r^2}\right)$$
$$T_2(r, \theta) = \left(r \cos\theta, r \sin\theta, -\sqrt{R^2 - r^2}\right)$$

with $r^2 \leq 1$ or $r \leq 1$ as the domain (θ goes in its natural range $0 \leq \theta \leq 2\pi$ when there is no restriction on it, recall).

We can also solve for r which will actually alleviate having to break the thing into two pieces. Getting $r = \sqrt{R^2 - z^2}$ we have, substituting,

$$T(\theta, z) = \left(\sqrt{R^2 - z^2} \cos\theta, \sqrt{R^2 - z^2} \sin\theta, z\right)$$

Here we have $-1 \leq z \leq 1$ and $0 \leq \theta \leq 2\pi$.

Finally, in spherical, this is the best: we have $\rho = R$ a constant. So just plugging in R directly we have

$$T(\varphi, \theta) = (R \sin\varphi \cos\theta, R \sin\varphi \sin\theta, R \cos\varphi).$$

For things like hemispheres, etc. we would vary φ from 0 to $\pi/2$ instead of 0 to π.

Example 2.1.4 (Cone). Consider the cone $z^2 = x^2+y^2$ with $z \geq 0$. In cylindrical this is $r = z$ so applying the same techniques as above we have $(z \cos\theta, z \sin\theta, z)$ with $z \geq 0$, and $0 \leq \theta \leq 2\pi$. But the cone also has a simpler description in spherical coordinates, $\varphi = \pi/4$. This gives another parameterization, $(\rho(\sqrt{2}/2) \cos\theta, \rho(\sqrt{2}/2) \sin\theta, \rho\sqrt{2}/2)$ since $\cos(\varphi) = \cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$.

Example 2.1.5 (Paraboloid). The paraboloid is given by $z = x^2 + y^2$. So therefore, of course, the parameterization $(x, y, x^2 + y^2)$ works. However, cylindrically we have $(r \cos\theta, r \sin\theta, r^2)$. We might ask ourselves, which representation is "better." They are equally good, but it really depends on what part of the paraboloid you want—what's the domain of integration going to be. For example if you want a domain circularly shaped, say it satisfies $x^2 + y^2 \leq R^2$, then cylindrical is better, since then the domain would be described by $r \leq R$ and $0 \leq \theta \leq 2\pi$) (remember the goal is to simplify bounds of integration, etc). On the other hand if you want to reduce it to a rectangle or triangle in the plane, the regular graph parameterization will do.

2.2 Example Line and Surface Integrals

Ok enough talk, time for action.

Problem 2.1. *Given a hemisphere S defined by $z = \sqrt{R^2 - x^2 - y^2}$ with $z \geq 0$, and $x^2 + y^2 \leq 1$, we have a mass density function $m(x, y, z) = x^2 + y^2$ on it. Find the total mass.*

Solution. The total mass is going to be, of course, $\iint_S (x^2 + y^2) dS$. Incidentally, this is also going to calculate the rotational inertia of a hemisphere of radius R of uniform density about the z-axis. Using spherical coordinates we have S is parameterized by $(R \sin\varphi \cos\theta, R \sin\varphi \sin\theta, R \cos\varphi)$ with $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi/2$ (note it's only up to $\pi/2$ because it's only the top half of the sphere). We find the integrand, when substituting this guy for x and y, is $\rho^2 \sin^2\varphi$ (because $\sin^2\theta + \cos^2\theta = 1$). So we have

$$\iint_S \rho^2 \sin^2\varphi \, dS$$

But now $dS = \|T_\varphi \times T_\theta\| \, d\varphi \, d\theta$ so we have to calculate the cross-product thingy. We have

$$T_\varphi = (R \cos\varphi \cos\theta, R \cos\varphi \sin\theta, -R \sin\varphi)$$
$$T_\theta = (-R \sin\varphi \sin\theta, R \sin\varphi \cos\theta, 0)$$

After a bit of tædium we find that

$$T_\varphi \times T_\theta = R^2 \sin\varphi(\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$$

and that it has magnitude $R^2 \sin\varphi$. Hence the total mass is going to be

$$\int_0^{2\pi} \int_0^{\pi/2} (R^2 \sin^2\varphi)(R^2 \sin\varphi) \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\pi/2} R^4 \sin^3\varphi \, d\varphi \, d\theta$$

and pulling R^4 out we have that the sine integral is one of the standard trigonometric integrals (use that $\sin^3\varphi = \sin\varphi(1 - \cos^2\varphi)$ and substitute $u = \cos\varphi$. I remind the dear reader that actual final evaluation of 1-dimensional integrals is not the important issue for this exam). □

Problem 2.2. *Evaluate the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}$ is the vector field $(2x, 2y, 2z)$ through the paraboloid $z = x^2 + y^2 + 1$, $x^2 + y^2 \leq 1$, where the normal points upward, away from the origin (i.e. calculate the flux of $\mathbf{F}$ thru a paraboloid).*

Solution. Note that unless the surface is *closed* (that is, has no edges, or has a clear inside and outside), the direction of the normal must be specified in the problem. If it is closed, convention is to take it to point outward. We will give an example of such a problem shortly. Now the paraboloid in this example is almost the paraboloid in the example in the previous section except z is shifted up by one. Since the domain is circular we will use cylindrical coordinates to arrive at $(r \cos\theta, r \sin\theta, r^2 + 1)$. We have two candidates for a normal vector, namely $T_r \times T_\theta$ or $T_\theta \times T_r$ which is just the negative of the other. The specification given is "pointing upward" so this means the one we want is the one with positive z-coordinate. So we have

$$T_r = (\cos\theta, \sin\theta, 2r)$$

and

$$T_\theta = (-r \sin\theta, r \cos\theta, 0)$$

After mucking around with cross products and maybe observing that $\sin^2\theta + \cos^2\theta = 1$, we find that $T_r \times T_\theta = (-2r^2\cos\theta, -2r^2\sin\theta, r)$. Since $r \geq 0$ this is the right one. Therefore $\mathbf{F} \cdot (T_r \times T_\theta)$ is what we need to integrate, over the domain $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$. The $\mathbf{F}$, in terms of r, θ, z is just done by substitution and is equal to $2(r\cos\theta, r\sin\theta, r^2 + 1)$. Dot-producing we have $-4r^3\cos^2\theta - 4r^3\sin^2\theta + 2r(r^2+1) = -2r^3 + 2r$. So our final formula for the flux is

$$\int_0^{2\pi}\int_0^1 (-2r^3 + 2r)\,dr\,d\theta$$

which yields $2\pi(-1/2 + 1) = \pi$. □

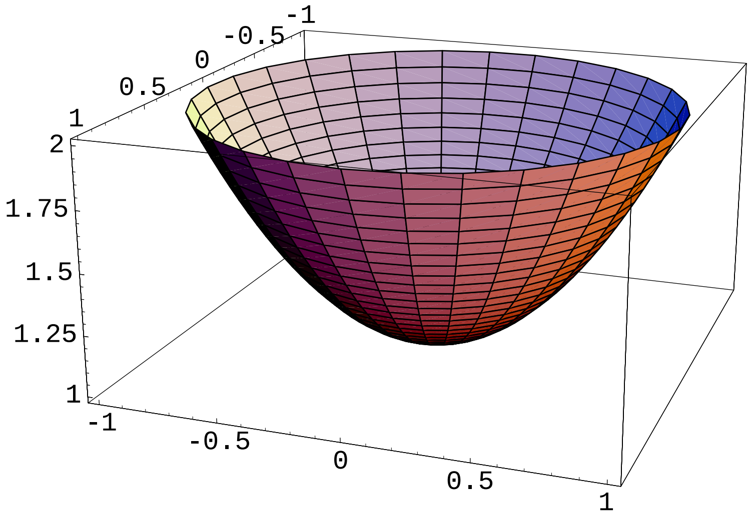

Figure 1: The Paraboloid

Problem 2.3. *Calculate the flux of the same vector field, except now through the* closed *surface given by the same paraboloid* plus *the top end-cap, the disk of radius 1, two units above the origin.*

Solution. The paraboloid has the same equation $(r\cos\theta, r\sin\theta, r^2 + 1)$. The top guy has the equation $z = 2$ with domain $x^2 + y^2 \leq 1$. This has the trivial parameterization $(x, y, 2)$ (it's a graph where the last guy is constant). But once again since we have circular domain, it is more useful to convert to cylindrical, giving $(r\cos\theta, r\sin\theta, 2)$ for it with r and θ having the same domain. The total flux is going to be the flux through the paraboloid plus the flux through this guy. However, the normal vector, by convention, is the outward-pointing. In particular, since the paraboloid is the "bottom side" of our closed surface, the normal has to point *downward*. So the flux for that is now $-\pi$, the negative of what we computed above. For the disk we take the vector pointing upward and away from the origin. Its vector is $(0, 0, r)$ as you can readily check. So the integral is going to be $\int_0^{2\pi}\int_0^1 4r\,dr\,d\theta = 4\pi$. Hence the total integral is $4\pi - \pi = 3\pi$. □