Solutions to Midterm 1

Chris Tiee

February 12, 2006

Problem 0.4. *Let f be a scalar field. The gradient ∇f of f is given by $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$.*

1. *Calculate $\nabla \times \nabla f$.*
2. *Calculate $\nabla \times y\mathbf{i}$.*
3. *Does there exist a scalar field f such that $\nabla f = y\mathbf{i}$?*

Solution.

1. Consider the standard Euclidean metric on the C^ω-manifold $M = \mathbb{R}^3$ with the standard differentiable structure given by the single analytic chart $(U = \mathbb{R}^3, \varphi = Id)$. We construct the de Rham complex on $\mathbb{R}^3$ and show that the gradient and the curl are given by the Hodge star operator. Let $\Omega^k(M)$ be the vector space of sections of the bundle of k-fold alternating tensors on M, spanned by the wedge products of the standard linear operators dx^1, dx^2, dx^3 where $dx^i(e_j) = \delta^i_j = 0$ if $i \neq j$, and 1 if $i = j$, and $e_1 = \mathbf{i}$, $e_2 = \mathbf{j}$, and $e_3 = \mathbf{k}$. The wedge products of the dx^i, $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ where $1 \leq i_1 < \cdots < i_k \leq 3$ form a basis for $\Lambda^k(\mathbb{R}^3)$, although you did not need to state this in order to receive full credit for this problem. We define the operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ as follows:

   $$(0.1)\quad d\omega(\xi_1, \dots, \xi_{k+1}) := \sum_{\ell=1}^{k+1} \sum_{j=1}^{3} \sum_{1\leq i_1 < \cdots < i_k \leq 3} (-1)^{\ell-1} \xi^j_\ell \frac{\partial \omega_{i_1 \dots i_k}}{\partial x^j} dx^{i_1} \wedge \cdots \wedge dx^{i_k}(\xi_1, \dots, \hat{\xi}_\ell, \dots, \xi_{k+1})$$

   where the $\omega_{i_1 \dots i_k}$ are the unique coefficients such that

   $$\omega = \sum_{1\leq i_1 < \cdots < i_k \leq 3} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

   and $\hat{\xi}_\ell$ means to omit the ℓth vector in the expression above. d is called the **exterior derivative on the Grassmann algebra** $\Omega^*(M)$.

   Theorem 0.4.1. *$d^2 = 0$, that is, $d \circ d : \Omega^k(M) \to \Omega^{k+2}(M)$ is the 0 operator.*

Proof. It is enough to check the operation of $d(d\omega)$ on a basis set $(e_{j_1}, \dots, e_{j_{k+2}})$:

$$(0.2)\quad d(d\omega)(e_{j_1}, \dots, e_{j_{k+2}}) =$$

$$\sum_{\ell=1}^{k+2} \sum_{j=1}^{3} \sum_{1 \le i_1 < \dots < i_{k+1} \le 3} (-1)^{\ell-1} e^j_{j_\ell} \frac{\partial (d\omega)_{i_1 \dots i_{k+1}}}{\partial x^j} dx^{i_1} \wedge \dots \wedge dx^{i_{k+1}} (e_{j_1}, \dots, \hat{e}_{j_\ell}, \dots, e_{j_{k+2}})$$

$$= \sum_{\ell=1}^{k+2} (-1)^{\ell-1} \frac{\partial (d\omega)_{j_1 \dots \hat{j}_\ell \dots j_{k+2}}}{\partial x^{j_\ell}}$$

$$= \sum_{1 \le p < \ell \le k+2} (-1)^{\ell+p} \frac{\partial^2 \omega_{j_1 \dots \hat{j}_p \dots \hat{j}_\ell \dots j_{k+2}}}{\partial x^{j_\ell} \partial x^{j_p}} + \sum_{1 \le \ell \le p \le k+1} (-1)^{\ell+p} \frac{\partial^2 \omega_{j_1 \dots \hat{j}_\ell \dots \hat{j}_{p+1} \dots j_{k+2}}}{\partial x^{j_\ell} \partial x^{j_{p+1}}}$$

$$= \sum_{1 \le p < \ell \le k+2} (-1)^{\ell+p} \frac{\partial^2 \omega_{j_1 \dots \hat{j}_p \dots \hat{j}_\ell \dots j_{k+2}}}{\partial x^{j_\ell} \partial x^{j_p}} + \sum_{1 \le \ell < q \le k+2} (-1)^{\ell+q-1} \frac{\partial^2 \omega_{j_1 \dots \hat{j}_\ell \dots \hat{j}_q \dots j_{k+2}}}{\partial x^{j_\ell} \partial x^{j_q}}$$

$$= \sum_{1 \le p < q \le k+2} \left(\frac{\partial^2 \omega_{j_1 \dots \hat{j}_p \dots \hat{j}_q \dots j_{k+2}}}{\partial x^{j_p} \partial x^{j_q}} - \frac{\partial^2 \omega_{j_1 \dots \hat{j}_p \dots \hat{j}_q \dots j_{k+2}}}{\partial x^{j_q} \partial x^{j_p}} \right) = 0$$

where we recursively substitute for the second $d\omega$ and juggle indices, taking into account the fact that all the indices get pushed up by one if ℓ is elided earlier than p, setting the index $q = p+1$, and recognizing symmetry. The last line follows, of course, from the equality of mixed partials. Therefore $d(d\omega)$ is the zero $(k+2)$-form. $\square$

To finish off the theorem, let $\mathfrak{X}(M)$ denote the set of all smooth vector fields on $\mathbb{R}^3$. We would like to show that the following diagram commutes:

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "\nabla"] \arrow[d, "\mathrm{Id}"] & \mathfrak{X}(M) \arrow[r, "\nabla\times"] \arrow[d, "g"] & \mathfrak{X}(M) \arrow[r, "\nabla\cdot"] \arrow[d, "\star\circ g"] & C^\infty(M) \arrow[d, "\mathrm{dvol}"] \\
\Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \Omega^3(M)
\end{tikzcd}$$

where g denotes the **metric dual identification** $(A, B, C) \mapsto Adx + Bdy + Cdz$ and $\star$ denotes the **Hodge Star Operator** defined by

$$\star(Adx + Bdy + Cdz) = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy.$$

and *vol* sends a scalar function to the function times the **volume form** on M given by $dx \wedge dy \wedge dz$ (we will see that in other coordinate systems, transforming the volume form gives us the Jacobian). Then by examining components we see that it is plain as day that g, $\star$ are invertible linear operators and $\nabla f = g^{-1}(df)$, and

$$(0.3)\quad d(F_1 dx + F_2 dy + F_3 dz)$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx \wedge dy$$

which shows $\nabla \times \mathbf{F} = g^{-1} \star^{-1} (d(g(\mathbf{F})))$. This shows the commutativity of the diagram.

Now, since $d^2 = 0$,

$$\nabla \times \nabla f = g^{-1} \star^{-1} d(g(g^{-1}(df))) = g^{-1} \star^{-1} (d^2 f) = g^{-1} \star^{-1} (0) = 0$$

which shows us that $\nabla \times \nabla f = 0$ for any scalar field f.

2. To compute this we just note that under metric duality, $y\mathbf{i}$ is identified with ydx and then taking d of it, we have $d(ydx) = dy \wedge dx$. Now the inverse of the Hodge star of this is $-dz$ (by the antisymmetry of the wedge product) which in turn is re-identified via metric dual $-\mathbf{k}$. Therefore $\nabla \times y\mathbf{i} = -\mathbf{k}$.

3. Most certainly not! We just computed by part (a) that the curl of any vector field that's the gradient of something must be zero. And by part (b) $y\mathbf{i}$ has the manifestly nonzero curl $-\mathbf{k}$.

□