

## Homework 1 Solution

MATH 20E

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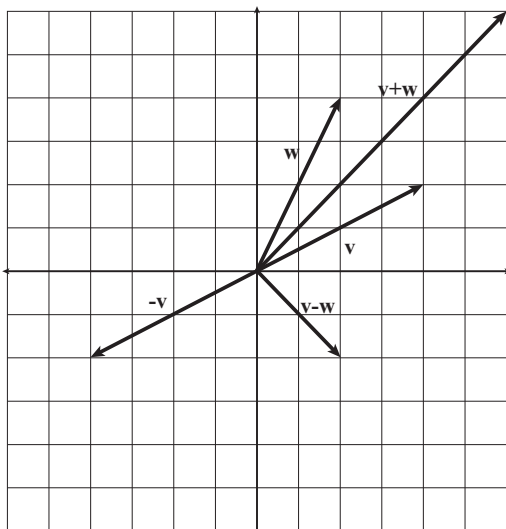
### 1 Chapter 1

#### 1.1 1.1 Problems 4, 5, 9, 14, 15

**Problem 1.1.4.**  $(2, 3, 5) - 4\mathbf{i} + 3\mathbf{j} = (-2, 6, 5)$ . Please do not mix  $\mathbf{i}$  and  $\mathbf{j}$  and comma notation like this in real life, you will confuse yourself.

**Problem 1.1.5.** Sketch  $\mathbf{v} = (2, 1)$  and  $\mathbf{w} = (1, 2)$ , their negatives, sum, and difference.

*Solution.*  $(3, 3)$  is their sum,  $(1, -1)$  is their difference.



□

**Problem 1.1.9.** What restrictions must be made on  $x$ ,  $y$ , and  $z$  so that the triple  $(x, y, z)$  will represent a point on the  $y$ -axis? On the  $z$ -axis? In the  $xz$ -plane? In the  $yz$ -plane?

*Solution.* The  $y$ -axis measures how far along the  $y$ -direction you are. Therefore the  $y$  is free to roam around, by definition. It's the other coordinates that must be restricted, namely  $x = 0$  and  $z = 0$ . Similarly to be on the  $z$ -axis, we require  $x = 0$  and  $y = 0$ . In the  $xz$ -plane, we now have two dimensions of freedom, namely both  $x$  and  $z$  are free to move around. That means the remaining coordinate is restricted, so  $y = 0$ . For the  $yz$ -plane,  $x = 0$ . □

**Problem 1.1.14.** Find a parameterization for the line thru  $(0, 2, 1)$  in the direction  $2\mathbf{i} - \mathbf{k}$ .

*Solution.* A line is determined by a point on it, and a direction vector. So they've already given you everything. Formally, however, to write it out in parameterized form, we have

$$\mathbf{r}(t) = (0, 2, 1) + t(2, 0, -1).$$

This means at time  $t$  you are at some location away from  $(0, 2, 1)$  by adding the increment of  $t$  times the vector  $(2, 0, -1)$ . Over all time, the path traced out by  $\mathbf{r}(t)$  is the line sought. □

**Problem 1.1.15.** Find a parameterization for the line thru  $(-1, -1, -1)$  and  $(1, -1, 2)$ .

*Solution.* Here it's not as straightforward as the last problem. You have various choices in the matter. It is important to know that "the" parameterization of a line is not unique. Either point will work; the matter is finding the direction vectors corresponding to your choice. If we choose  $(-1, -1, -1)$  as the starting point, then taking  $\mathbf{v} = (1, -1, 2) - (-1, -1, -1) = (2, 0, 3)$  we have  $\mathbf{r}_1(t) = (-1, -1, -1) + t(2, 0, 3)$ . We could just as easily use  $(-2, 0, 3)$  for it so we could use  $\mathbf{r}_2(t) = (-1, -1, -1) + t(-2, 0, -3)$ . Taking the other point  $(1, -1, 2)$  and the same direction vectors we get the parameterizations  $\mathbf{r}_3(t) = (1, -1, 2) + t(2, 0, 3)$  and  $\mathbf{r}_4(t) = (1, -1, 2) + t(-2, 0, -3)$  which also work. Heck, even  $\mathbf{r}_5(t) = (-1, -1, -1) + (t^5 + 3t^4 - 3\sin^2(t) + 1)(-2, 0, -3)$  will work.  $\square$

## 1.2 Section 2, Problems 10, 15, 17

The dot (inner) product is actually a very important geometrical operation. Perhaps *the* most important concept you will learn in this course (or did learn in 20C). Generalized to higher dimensions, the dot product is taken as the basis for defining lengths and angles in higher-dimensional spaces, where you can't see anything. It embodies the concepts of "independence," "dimension," and "breaking things up into nice components." Take care to master it.

**Problem 1.2.10.** Find  $\|\mathbf{u}\|$ ,  $\|\mathbf{v}\|$  and  $\mathbf{u} \cdot \mathbf{v}$  for  $\mathbf{u} = -\mathbf{i} + 3\mathbf{k}$  and  $\mathbf{v} = 4\mathbf{j}$ .

*Solution.*  $\|\mathbf{u}\| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$ ,  $\|\mathbf{v}\| = 4$  (it is very easy to take the length of a vector that has only one component!). For  $\mathbf{u} \cdot \mathbf{v}$  we write them out in components  $(-1, 0, 3)$  and  $(0, 4, 0)$ . A quick glance reveals that each number gets multiplied by 0 somewhere, so we have  $\mathbf{u} \cdot \mathbf{v} = 0$ .  $\square$

**Problem 1.2.15.** Find the projection of  $\mathbf{v} = (2, 1, -3)$  onto  $\mathbf{u} = (-1, 1, 1)$ .

*Solution.* We make use of the formula  $\mathbf{Proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{w}(\mathbf{w} \cdot \mathbf{v})$  with  $\mathbf{w} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$ . Now  $\mathbf{w} = \frac{1}{\sqrt{3}}(-1, 1, 1)$  since  $u$  has length 3. Therefore  $\mathbf{Proj}_{\mathbf{u}}(\mathbf{v}) = \frac{1}{3}(-2 + 1 - 3)(-1, 1, 1) = \left(\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$ , the square root being eliminated because the factor appears twice (picture to be inserted here).  $\square$

**Problem 1.2.17.** Find two nonparallel vectors both orthogonal to  $(1, 1, 1)$

*Solution.* This is a good problem, as it has inklings of the concept of *linear independence* from linear algebra, which you may encounter in your future mathematical wanderings (Math 20F and many upper division classes). Here translate orthogonality into the language of dot products to set up an equation: find vector solutions to the equation  $\mathbf{v} \cdot (1, 1, 1) = 0$ . But  $\mathbf{v}$  is determined by its components, say,  $(x, y, z)$ . So this gives us the linear equation

$$x + y + z = 0.$$

Since this is only one equation with 3 unknowns, it can be expected to have many solutions. Indeed looking at a picture, all vectors orthogonal to  $(1, 1, 1)$  form a plane. How to solve this? Plug in easy values for  $y$  and  $z$  and see what it gives for  $x$ . The first impulse is to choose  $y = z = 0$ , but that forces  $x = 0$  which is bad since  $(0, 0, 0)$  is parallel to everything. Try  $y = 0$  and  $z = -1$ , this gives  $x = 1$ . So taking these 3 together we get  $(1, 0, -1)$ . To get another one, plug in something different, say  $y = -1$  and  $z = 0$ . This gives  $x = 1$  again, so another vector is  $(1, -1, 0)$ . To see that these aren't parallel, divide corresponding components and see if they give the same number; if all 3 ratios are the same, they're parallel. Otherwise not. We see the ratio of the first two is 1 and the second is 0, so this immediately tells us these aren't parallel.

Note that there are (infinitely) many other different answers to this problem, depending on which values you pick for two of the variables in the above linear equation.  $\square$

### 1.3 Section 3, Problems 3, 10, 13, 25, 26

The cross product is a very strange beast. It's very hard to say quite what it is. It derives from physics and has much to do with rotations (not surprising as it is the Lie bracket for the rotation group  $SO(3)$ , but that, my dear reader, is another story), magnetic fields, and the like. It only exists in 3 dimensions, and I know of one professor who refuses to teach it, because of the fact that it doesn't generalize (nonsense; there are plenty of Lie brackets and 2nd-rank antisymmetric tensor densities that need wedging together, but that, again, is another story). If the reader cares, there is a nice exposition on the history of these things in the book on pages 54-61. I am not quite sure if it was worth the large number of extra trees. Anyway, I hate computing these things. If you're not fond of the rather cute determinant formula given in the book, you can use the following trick, if you're not afraid of matrix multiplication:

Given  $\mathbf{v} = (v_x, v_y, v_z)$  and  $\mathbf{w} = (w_x, w_y, w_z)$  we have

$$\mathbf{v} \times \mathbf{w} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

(we identify vectors written vertically and horizontally). To remember this, note that there are 0's on the diagonal of the matrix, and it is antisymmetric (equal to minus its transpose). The vector  $\mathbf{v}$  gets jammed into the lower left corner and put in backwards; and middle gets the minus sign (as in the determinant formula). Finally re-copy with opposite signs in the upper right (thanks to D.H. Chang for teaching me this one in Electromagnetics).

**Problem 1.3.3.** Calculate  $\mathbf{a} \times \mathbf{b}$  where  $\mathbf{a} = (1, -2, 1)$  and  $\mathbf{b} = (2, 1, 1)$ .

*Solution.*

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = (-2 \cdot 1 - 1 \cdot 1)\mathbf{i} - (1 \cdot 1 - 2 \cdot 1)\mathbf{j} + (1 \cdot 1 + 2 \cdot 2)\mathbf{k} = -3\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

or

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \cdot 1 - 2 \cdot 1 \\ 1 \cdot 2 - 1 \cdot 1 \\ 2 \cdot 2 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} = -3\mathbf{i} + \mathbf{j} + 5\mathbf{k}.$$

□

**Problem 1.3.10.** Find all unit vectors orthogonal to both  $(-5, 9, -4)$  and  $(7, 8, 9)$ .

*Solution.* First off, all vectors orthogonal to both of those vectors lie on a line, i.e. they are all scalar multiples of each other (this is because they are non-parallel and lie in a plane). Therefore there are two possible unit vectors: one obtained by normalizing any vector orthogonal to both, and its negative (since multiplying by -1 doesn't change the magnitude of a vector). Those are *all* the possibilities. Now how do we find a vector orthogonal to both? That's what we have the cross product for:

$$(-5, 9, -4) \times (7, 8, 9) = \begin{bmatrix} 0 & 4 & 9 \\ -4 & 0 & 5 \\ -9 & -5 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \cdot 8 + 9 \cdot 9 \\ -4 \cdot 7 + 5 \cdot 9 \\ -9 \cdot 7 - 5 \cdot 8 \end{bmatrix} = \begin{bmatrix} 113 \\ 17 \\ 103 \end{bmatrix}$$

which has got to be the most absurd-looking vector you've ever seen. Its length is  $\sqrt{23667}$ . So therefore the two unit vectors are  $\frac{1}{\sqrt{23667}}(113, 17, 103)$  and  $-\frac{1}{\sqrt{23667}}(113, 17, 103)$  □

**Problem 1.3.13.** Find  $\mathbf{u} + \mathbf{v}$ ,  $\mathbf{u} \cdot \mathbf{v}$ ,  $\|\mathbf{u}\|$ ,  $\|\mathbf{v}\|$ , and  $\mathbf{u} \times \mathbf{v}$  for  $\mathbf{u} = (1, -2, 1)$  and  $\mathbf{v} = (2, -1, 2)$ .

*Solution.* Boy they really want to test whether you know your vectorial operations or not.

a.  $\mathbf{u} + \mathbf{v} = (3, -3, 3)$

b.  $\mathbf{u} \cdot \mathbf{v} = 1 \cdot 2 + (-2) \cdot (-1) + 1 \cdot 2 = 9$

c.  $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

d.  $\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$ , and finally

e.

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \cdot -1 - 2 \cdot 2 \\ 1 \cdot 2 - 1 \cdot 2 \\ 2 \cdot 2 - 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$$

□

**Problem 1.3.25.** Find an equation for the plane that passes through the point  $(1, 2, -3)$  and is perpendicular to the line  $\mathbf{v}(t) = (0, -2, 1) + t(1, -2, 3)$ .

*Solution.* The thing to remember about lines and planes is that lines are determined by a vector *parallel* to it, while planes are determined by a vector *perpendicular* to it. This is the single most common mistake I saw in 20C, confusing the equations and directionality of lines and planes. In this problem, we have to find a plane *perpendicular* to a line with tangent vector  $(1, -2, 3)$ . But since  $(1, -2, 3)$  is parallel to the line, it will be perpendicular to the plane we're looking for. Therefore we can use this vector to as the normal vector that defines the plane. Taking  $\mathbf{n} = (1, -2, 3)$  the point on the plane,  $\mathbf{r}_0 = (1, 2, -3)$ , and letting  $\mathbf{r} = (x, y, z)$  we find the equation  $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ :

$$(1, -2, 3) \cdot ((x, y, z) - (1, 2, -3)) = (1, -2, 3) \cdot (x - 1, y - 2, z + 3) = x - 1 - 2(y - 2) + 3(z + 3)$$

Distributing out and setting equal to 0 we have

$$x - 2y + 3z + 12 = 0$$

as the equation that defines the plane. □

**Problem 1.3.26.** Find the equation of the line that passes through the point  $(1, -2, -3)$  and is perpendicular to the plane  $3x - y - 2z + 4 = 0$ .

*Solution.* This problem is the inverse of the previous one. The same directionality rules apply. Since the line we're looking for is perpendicular to the plane, the plane's normal vector will be along the direction of this line. Hence we take this vector as the direction vector. We can find the normal by taking corresponding coefficients in the equation defining the plane:  $3x - 1y - 2z + 4 = 0$  gives us  $(3, -1, -2)$ . (Why? If you've read ahead, the reason for this is because it is the gradient vector of  $3x - y - 2z$  and the gradient is perpendicular to level surfaces; the plane is just the 0 level surface of this function). Therefore taking the given point  $(1, -2, -3)$  on the line we let  $\mathbf{r}(t) = (1, -2, -3) + t(3, -1, -2)$  define our line. □