Homework 2 Solution

MATH 20E
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1 Chapter 1

1.1 Section 4, Problems 1, 4, 8, 12, 13

Problem 1.1.1. *Convert some points from cylindrical coordinates to rectangular and spherical coordinates.*

Solution. This should be reasonably straightforward plugging and chugging; note that the conversion from cylindrical to spherical uses formulas that are almost the same as rectangular to cylindrical:

$$\rho = \sqrt{r^2 + z^2}$$

$$\varphi = \tan^{-1}\left(\frac{r}{z}\right)$$

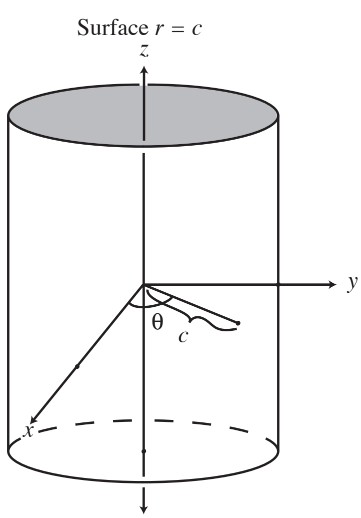
with the usual caveats of tan^{-1} sign conventions and adding nultiples of π applying (try to draw yourself a picture to determine what θ or ϕ is from context). It is also possible to convert to rectangular first, and then convert to spherical, though the formulas are more complicated. □

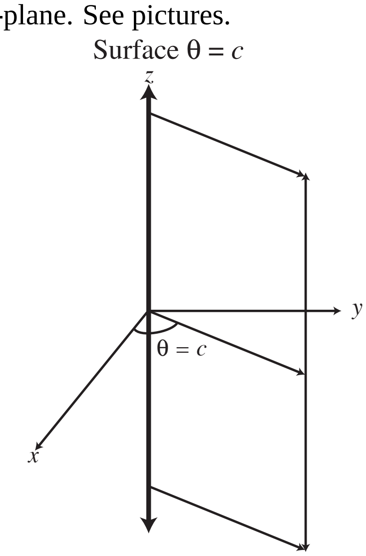
Problem 1.1.4.

1. *Describe the surfaces $r = const.$, $\theta = const.$, and $z = const.$*
2. *Describe the surfaces $\rho = const.$, $\theta = const.$, and $\varphi = const.$*

Solution. This is a good problem, because it illustrates how coordinate systems are actually defined—a point is determined by the intersection of the surfaces on which the coordinates are constant.

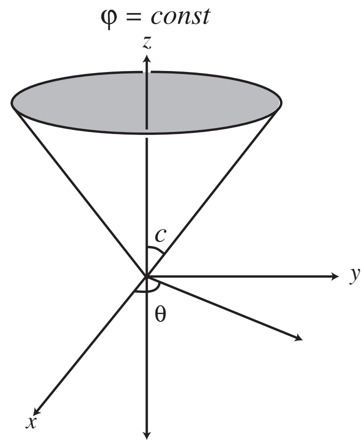
1. r = const. is a cylinder around the z-axis. This is how cylindrical coordinates gets its name. θ = const. is a half-plane, making a constant angle with the xz-plane. See pictures.





Note that it is a half-plane, namely it has an edge on the z-axis, because we only allow $r \geq 0$. The other "half" of the plane corresponds to $\theta = c + \pi$, a 180° turnaround. z = const. is a plane parallel to the xy-plane (easy to see).

2. ρ = const. is a sphere, θ = const is exactly the same as in (a). φ = const. is a cone with vertex at the origin and sides making a constant angle with the z-axis. If the constant is $< \pi/2$ then the cone opens upward, $> \pi/2$ opens downward, and $= \pi/2$ is the xy-plane.



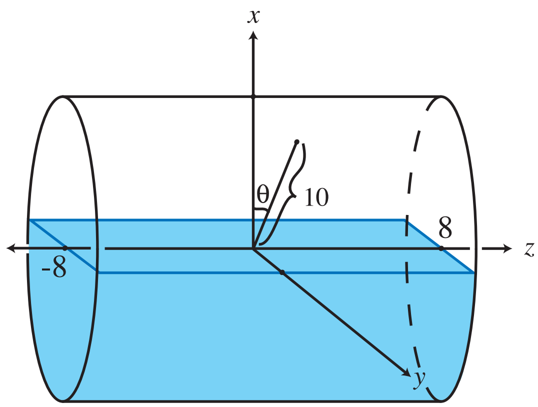
Graphics brought to you by the miracle of *Mathematica* and Adobe Illustrator 11. □

Problem 1.1.8. *Express the plane $z = x$ in cylindrical and spherical coordinates.*

Solution. Direct substitution gives $z = r\cos\theta$ in cylindrical coordinates. In spherical coordinates it's not as pretty, we have $\rho\cos\varphi = \rho\sin\varphi\cos\theta$, or canceling, just $\cos\varphi = \sin\varphi\cos\theta$. I don't know of any simpler-looking answer except maybe $\tan\varphi\cos\theta = 1$, but that has its problems of being undefined at $\varphi = \pi/2$. □

Problem 1.1.12. *A tank in the shape of a right circular cylinder of radius 10 ft and height 16 ft is half-filled and lying on its side. Describe the air space inside the tank by "suitably chosen" cylindrical coordinates.*

Solution. There are several solutions to this problem, depending on what one means by "suitable." My guess is that the most "suitable" would have the z-axis along the cylinder's axis. From there the x- and y-axes can be freely rotated (which will affect the θ range, and the origin can be slid along the z-axis, which affects the z range. This is what I did in section: choosing the z-axis to point right and x-axis to point straight up, this makes, by the right-hand rule, the y-axis to point out of the plane of the page. The air space describes the upper half, so this makes θ vary from $-\pi/2$ to $\pi/2$, or $3\pi/2$ to 2π and 0 to $\pi/2$. If x is directed out of the page, then y points down, and θ now varies from π to 2π or $-\pi$ to 0. r varies from 0 to 10 as the cylinder has radius 10. Choosing the origin to be exactly at the center of the cylinder makes z vary from -8 to 8. If the origin is on the left end-cap it should instead be 0 to 16. See picture below.



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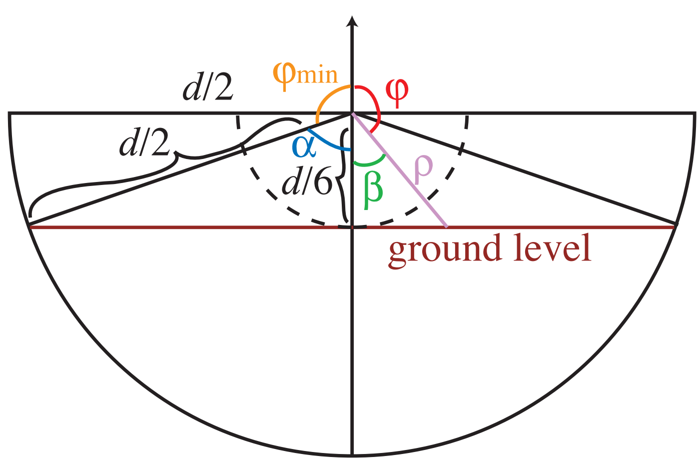
Problem 1.1.13. *A sphere of diameter* d *is to be buried a distance* $d/3$ *into the ground. Describe the buried portion in spherical coordinates.*

Solution.



We choose the origin to be the center of the sphere, so that the ground is at $z = -d/6$ from it. Other solutions are possible but are not as pretty. Since the buried portion still goes all the way around, we can immediately say that $0 \leq \theta \leq 2\pi$. Now what is left? We can take a 2D cross section, since we've taken care of θ already. There are two ways to proceed from this point: find the range of φ first, or ρ first. It is possible to get two different, but equivalent descriptions at this point: either make ρ look simple or φ look simple, i.e. for one of the coordinates we can have the simple-minded description that it is in between two constants. But it will force the other coordinate to have a not-so-simple description—the "simple" range of one coordinate will restrict the range of the other. This is because the "ground" makes φ and ρ depend on each other—they are forced to interact because a plane has a non-trivial equation in spherical coordinates. We do it both ways.

First is the φ-simple description (we did this in class). Now we find the range of φ: the maximum extent is clearly π. To find the minimum extent of φ (the angle $\varphi_{\min}$ in orange in the diagram above), we draw a radius from the origin to the point where the ground meets the circle. We look at α (in blue) which is $\pi - \varphi_{\min}$, and simple trigonometry tells us that $\frac{d}{6} = \frac{d}{2}\cos(\alpha)$. Shuffling around the numbers we find that $\alpha = \cos^{-1}\left(\frac{1}{3}\right)$. This is not any particularly nice angle (it's about 71° if you're interested), and so therefore the minimum extent of φ is $\pi - \cos^{-1}\left(\frac{1}{3}\right)$. Therefore we have

$$\pi - \cos^{-1}\left(\frac{1}{3}\right) \leq \varphi \leq \pi.$$

The maximum extent of ρ is cearly $d/2$ since the straight line, where the trouble is, is at most $d/2$ as well. The minimum extent is the straight line itself. We have drawn a typical such ρ in lavender. We want to see how this depends on φ, the red angle. Taking the green angle $\beta = \pi - \varphi$ for convenience, the same old trigonometry shows that $\frac{d}{6} = \rho\cos(\beta)$. This time we solve for ρ, so that $\rho = \frac{d}{6}\sec(\beta)$. Now plugging in $\beta = \pi - \varphi$ we find $\rho = -\frac{d}{6}\sec(\varphi)$ so $-\frac{d}{6}\sec(\varphi) \leq \rho \leq \frac{d}{2}$. So the total description is:

$$-\frac{d}{6}\sec(\varphi) \leq \rho \leq \frac{d}{2}$$
$$\pi - \cos^{-1}\left(\frac{1}{3}\right) \leq \varphi \leq \pi$$
$$0 \leq \theta \leq 2\pi$$

For the second way of doing this, taking ρ first, we easily see the minimum extent of ρ is $\frac{d}{6}$ and maximum is $\frac{d}{2}$. For φ the maximum is again π because the vertical is included. We again refer to the ρ and β in the diagram above, and we get the same relation $\frac{d}{6} = \rho\cos(\beta) = -\rho\cos(\varphi)$. Now solving for φ instead, we have $\varphi = \cos^{-1}\left(-\frac{d}{6\rho}\right)$. So therefore our complete description is

$$\frac{d}{6} \leq \rho \leq \frac{d}{2}$$
$$\cos^{-1}\left(-\frac{d}{6\rho}\right) \leq \varphi \leq \pi$$
$$0 \leq \theta \leq 2\pi$$

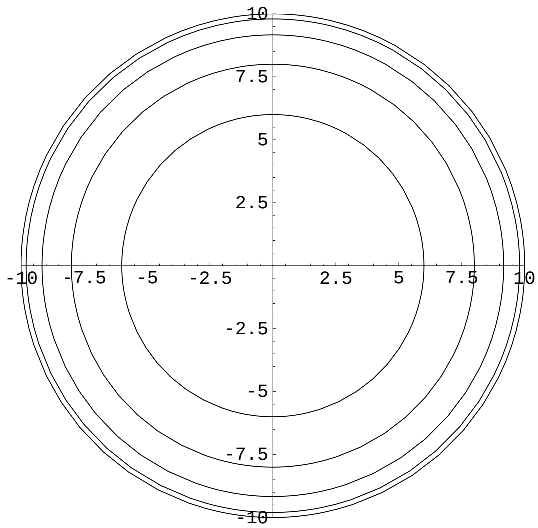
(actually this one looks a little prettier as we don't have to worry about what the heck $\cos^{-1}\left(\frac{1}{3}\right)$ is...). Looking at this, though, many a student may wonder, what is wrong with taking $\frac{d}{6} \leq \rho \leq \frac{d}{2}$ and $\pi - \cos^{-1}\left(\frac{1}{3}\right) \leq \varphi \leq \pi$. together? After all, "minimum extent" means minimum extent, right? And both ρ and φ achieve that those two minimum numbers. The trouble is, we can't achieve the true minimum extent of both ρ and φ *at the same time*. And when they can be minimum is dictated by the relation between ρ and φ. In fact, if we insist on allowing that, then the region we describe is really the area between the circle $\rho = \frac{d}{2}$ and the smaller circle shown as a dashed line. Do you see why? Ok! That was a very hard problem. Time to move on to bigger and better things. □

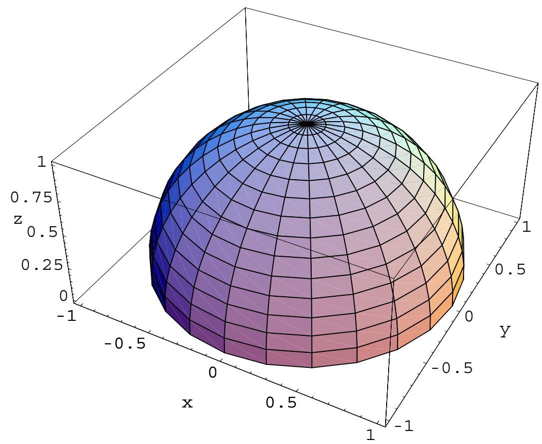
2 Chapter 2

2.1 Section 1, Problems 5, 8, 17

Problem 2.1.5. *Draw the level curves of* $f(x,y) = \sqrt{100 - x^2 - y^2}$ *with* c *ranging from* 0 *to* 10*, incrementing by* 2*. Sketch a graph.*

Solution. The graph of this is an upper hemisphere of radius 10. We rearrange $\sqrt{100 - x^2 - y^2} = c$ to $x^2 + y^2 = 100 - c^2$. Hence is a collection of circles, of radii 10, $\sqrt{96}$, $\sqrt{84}$, 8, 6, and the "degenerate" circle of radius 0, namely a point.



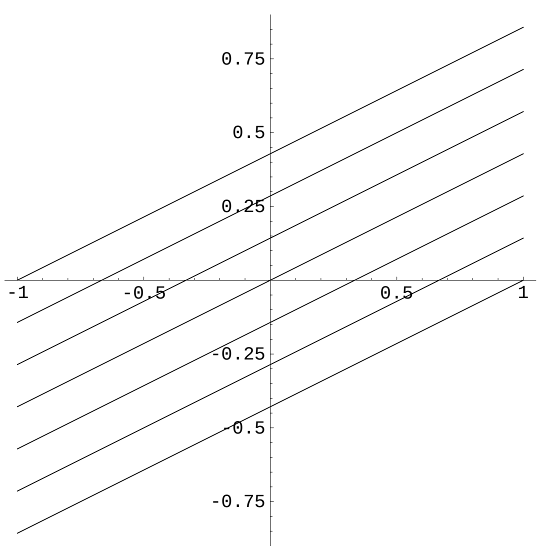


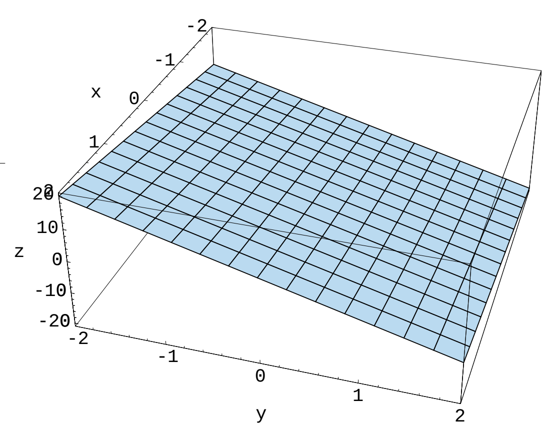
□

Problem 2.1.8. *Draw the level curves of* $f(x, y) = 3x - 7y$ *with* c *ranging from* -3 *to* 3, *incrementing by* 1. *Sketch a graph*

Solution. Solving $3x - 7y = c$ for y in terms of x we have $7y = 3x - c$ or $y = \frac{3}{7}x - \frac{c}{7}$ or basically lines with slope $\frac{3}{7}$ and shifted up and down along the y-axis by increments of $\frac{1}{7}$.

The graph can be estimated by looking at which way is the direction of increase—as c increases the level curves move down a little bit, so the general direction of increase is down and to the right. Realizing the function is linear (so the graph has to be a plane) and that level curves are slices of this graph, we have a plane tilted so that it increases while coming at you (see picture).





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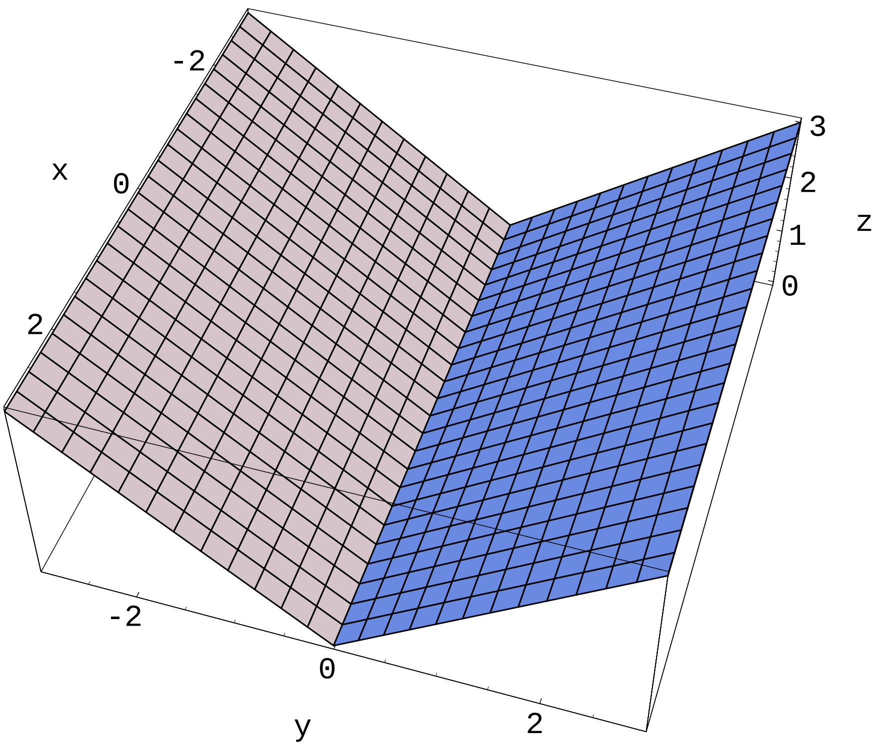
Problem 2.1.9. *Describe the level sets and graph of the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = |y|$.

Solution. This doesn't depend on x so it's the graph of $z = |y|$ in the yz-plane and this graph sweeps a surface out on the x-axis. It looks like a horse trough. The level curves $|y| = c$ are simply pairs of horizontal lines in the xy-plane if $c > 0$, the single line $y = 0$ if $c = 0$ and nothing at all if $c < 0$. Note that the level curves as a collection looks exactly like $f(x, y) = y$. This is where the labeling of the level curves is necessary to distinguish things like this.



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2.2 Section 3, Problems 1(ab), 2(b), 6(ab)

Problem 2.2.1. *Find* $\partial f/\partial x$ *and* $\partial f/\partial y$ *for*

a. $f(x, y) = xy$

b. $f(x, y) = e^{xy}$

Solution.

a. We just hold y constant for $\partial f/\partial x$ so that $\partial f/\partial x = y$. Similarly $\partial f/\partial y = x$.

b. A straightforward computation (recalling that $d/dt(e^{at}) = ae^{at}$) gives $\partial f/\partial x = ye^{xy}$ and $\partial f/\partial y = xe^{xy}$.

□

Joke 2.2.1. Two functions, e^x and a constant function are in a bar. They're chattin' it up rather nicely, when a really hot differential operator walks in. The constant function gets all scared and says "Oh damn, I gotta go... I'll catch up with ya later, ok?"

"Why?" asks e^x, rather irked.

"Well, you know… I don't wanna get reduced to 0," replies the constant, and quickly scuttles off.

Meanwhile, e^x grumbles "Pffft. Constant functions. What a bunch of wussies." So e^x, all cocky now, walks over to the operator, proudly announcing "YO! WASSUP MAH DIFFERENTIAL BABE?! DIZZAAYMN, LOOKIN' HOT, THERE! I'm e^x, yeah, that's right, e^x, and it *don't* get much *smooooother* than me!!"

The operator calmly retorts, "Hey… I'm $\partial/\partial y$."

Problem 2.2.2. *Find the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ for $z = \log\sqrt{1+xy}$.*

Solution. There's a possible ambiguity here, because I understand log here to be the natural log, but many may well interpret it as log to the base 10. Logarithms to the base 10 are a relic of the (bad) old days of the slide rule and having to calculate via tables. Anyhow, if you have calculated it as log to the base 10, which will introduce an extra factor of $\frac{1}{\ln 10}$, that will be an acceptable answer for this assignment. We have

$$\frac{\partial}{\partial x}\log\sqrt{1+xy} = \frac{1}{2}\frac{\partial}{\partial x}\log(1+xy) = \frac{1}{2(1+xy)}\cdot\frac{\partial(xy)}{\partial x} = \frac{y}{2(1+xy)}$$

(We save an extra gnarly computation involving the chain rule by remembering $\log(a^b) = b\log a$ and that $\sqrt{a} = a^{1/2}$. By symmetry and exchanging x and y, we have

$$\frac{\partial z}{\partial y} = \frac{x}{2(1+xy)}$$

□

Problem 2.2.6. *Compute the equation of the plane tangent to the graphs of (a) $f(x,y) = xy$ at $(0,0)$ and (b) $f(x,y) = e^{xy}$ at $(0,1)$.*

Solution. We make use of the formula $z = f(x_0,y_0) + \nabla f(x_0,y_0)\cdot(x-x_0, y-y_0)$ for the tangent plane.

a. $\nabla f(x,y) = (y,x)$ so evaluated at $(0,0)$ gives $(0,0)$. Similarly, $f(0,0) = 0$ so the tangent plane is the xy-plane, $z = 0$.

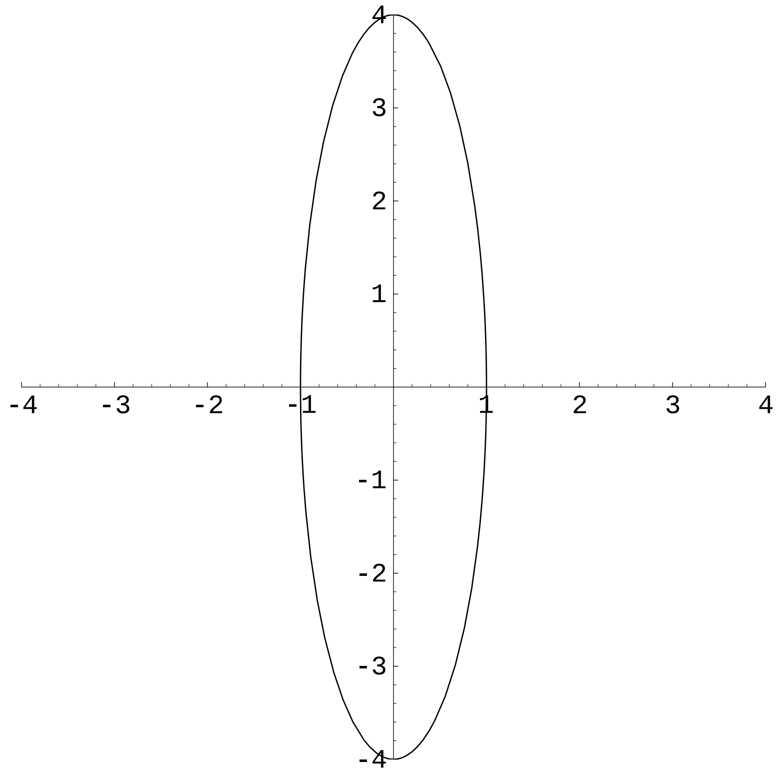
b. $\nabla f(x,y) = (ye^{xy}, xe^{xy}) = (1,0)$ at $(0,1)$ and $f(0,1) = 1$. Therefore the equation of the tangent plane is $z = 1 + x$.

□

2.3 Section 4, Problems 1, 3

Problem 2.3.1. *Plot the curve $x = \sin(t)$, $y = 4\cos(t)$, for $t \in [0, 2\pi]$.*

Solution. This is just a parameterization of the circle except really stretched in the y-direction.



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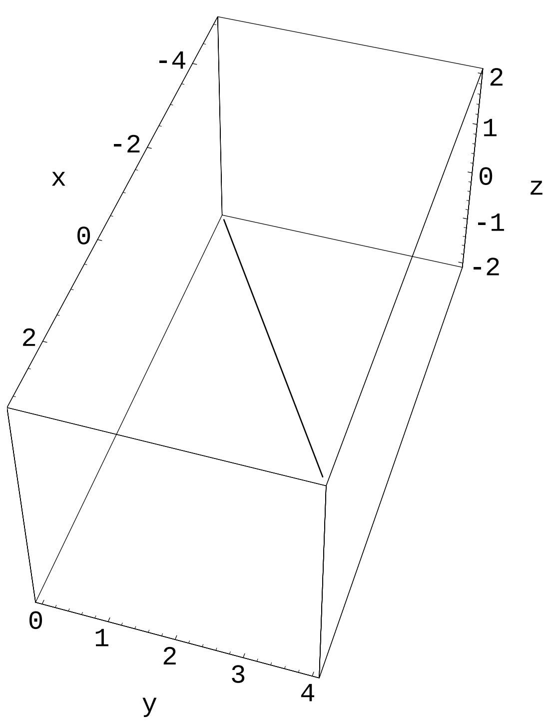
Problem 2.3.2. *Plot the curve* $\mathbf{c}(t) = (2t - 1, t + 2, t)$

Solution. This is just a straight line, which can be seen by separating out the constants: $\mathbf{c}(t) = (-1, 2, 0) + t(2, 1, 1)$.



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