Homework 2 Solution

MATH 20E
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1 Chapter 1

1.1 Section 4, Problems 1, 4, 8, 12, 13

Problem 1.1.4.

1. *Describe the surfaces* $r = \text{const.}$, $\theta = \text{const.}$, *and* $z = \text{const.}$
2. *Describe the surfaces* $\rho = \text{const.}$, $\theta = \text{const.}$, *and* $\varphi = \text{const.}$

Solution.

1. $r = \text{const.}$ is a cylinder around the z-axis. $\theta = \text{const.}$ is a half-plane, making a constant angle with the xz-plane, and $z = \text{const.}$ is a plane parallel to the xy-plane.
2. $\rho = \text{const.}$ is a sphere, $\theta = \text{const}$ is exactly the same as in (a), and $\varphi = \text{const.}$ is a cone with vertex at the origin and sides making a constant angle with the z-axis. If the constant is $< \pi/2$ then the cone opens upward, $> \pi/2$ opens downward, and $= \pi/2$ is the xy-plane.

□

Problem 1.1.8. *Express the plane* $z = x$ *in cylindrical and spherical coordinates.*

Solution. Direct substitution gives $z = r\cos\theta$ in cylindrical coordinates. In spherical coordinates it's not as pretty, we have $\rho\cos\varphi = \rho\sin\varphi\cos\theta$, or canceling, just $\cos\varphi = \sin\varphi\cos\theta$. I don't know of any simpler-looking answer except maybe $\tan\varphi\cos\theta = 1$, but that has its problems of being undefined at $\varphi = \pi/2$. □

Problem 1.1.12. *A tank in the shape of a right circular cylinder of radius 10 ft and height 16 ft is half-filled and lying on its side. Describe the air space inside the tank by "suitably chosen" cylindrical coordinates.*

Solution. There are several solutions to this problem, depending on what one means by "suitable." My guess is that the most "suitable" would have the z-axis along the cylinder's axis. From there the x- and y-axes can be freely rotated (which will affect the θ range, and the origin can be slid along the z-axis, which affects the z range. This is what I did in section: choosing the z-axis to point right and x-axis to point straight up, this makes, by the right-hand rule, the y-axis to point out of the plane of the page. The air space describes the upper half, so this makes θ vary from $-\pi/2$ to $\pi/2$, or $3\pi/2$ to 2π and 0 to $\pi/2$. If x is directed out of the page, then y points down, and θ now varies from π to 2π or $-\pi$ to 0. r varies from 0 to 10 as the cylinder has radius 10. Choosing the origin to be exactly at the center of the cylinder makes z vary from -8 to 8. If the origin is on the left end-cap it should instead be 0 to 16. □

Problem 1.1.13. *A sphere of diameter* d *is to be buried a distance* $d/3$ *into the ground. Describe the buried portion in spherical coordinates.*

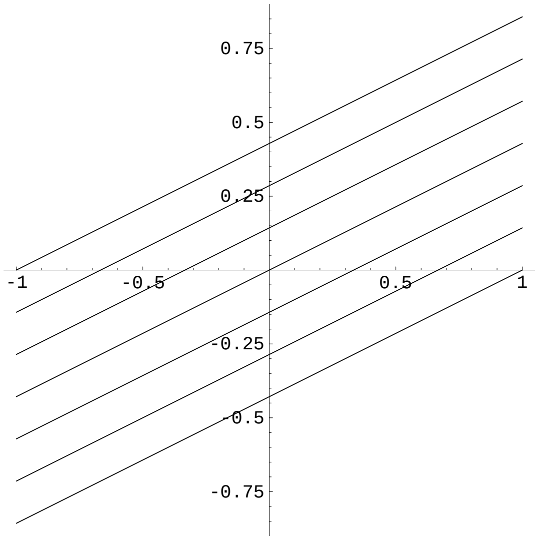
Solution. Note that the answer in the book relies on choosing the origin to be the center of the sphere, so that the ground is at $z = -d/6$ from it. Other solutions are possible but are not as pretty. □

2 Chapter 2

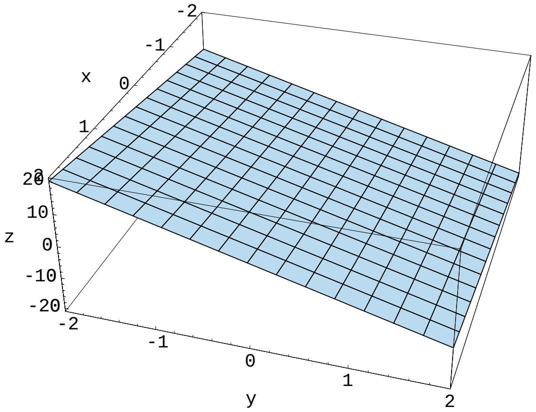
2.1 Section 1, Problems 5, 8, 17

Problem 2.1.8. *Draw the level curves of* $f(x,y) = 3x - 7y$ *with* c *ranging from* -3 *to* 3, *incrementing by* 1. *Sketch a graph*

Solution. Solving $3x - 7y = c$ for y in terms of x we have $7y = 3x - c$ or $y = \frac{3}{7}x - \frac{c}{7}$ or basically lines with slope $\frac{3}{7}$ and shifted up and down along the y-axis by increments of $\frac{1}{7}$.



The graph can be estimated by looking at which way is the direction of increase—as c increases the level curves move down a little bit, so the general direction of increase is down and to the right. Realizing the function is linear (so the graph has to be a plane) and that level curves are slices of this graph, we have a plane tilted so that it increases while coming at you (see picture).



□

2.2 Section 3, Problems 1(ab), 2(b), 6(ab)

Problem 2.2.1. *(to be inserted for full solution)*

Problem 2.2.2. *Find the partial derivatives* $\partial z/\partial x$ *and* $\partial z/\partial y$ *for* $z = \log\sqrt{1+xy}$.

Solution. There's a possible ambiguity here, because I understand log here to be the natural log, but many may well interpret it as log to the base 10. Therefore an answer with an extra factor of $\frac{1}{\ln 10}$ will be acceptable. We have

$$\frac{\partial}{\partial x}\log\sqrt{1+xy} = \frac{1}{2}\frac{\partial}{\partial x}\log(1+xy) = \frac{1}{2(1+xy)}\cdot\frac{\partial(xy)}{\partial x} = \frac{y}{2(1+xy)}$$

(We save an extra gnarly computation involving the chain rule by remembering $\log(a^b) = b\log a$ and that $\sqrt{a} = a^{1/2}$. By symmetry and exchanging x and y, we have

$$\frac{\partial z}{\partial y} = \frac{x}{2(1+xy)}$$

□

Joke 2.2.1. Two functions, e^x and a constant function are in a bar. They're chattin' it up rather nicely, when a really hot differential operator walks in. The constant function gets all scared and says "Oh damn, I gotta go… I'll catch up with ya later, ok?"

"Why?" asks e^x, rather irked.

"Well, you know… I don't wanna get reduced to 0," replies the constant, and quickly scuttles off.

Meanwhile, e^x grumbles "Pffft. Constant functions. What a bunch of wussies." So e^x, all cocky now, walks over to the operator, proudly announcing "YO! WASSUP MAH DIFFERENTIAL BABE?! DIZZAAYMN, LOOKIN' HOT, THERE! I'm e^x, yeah, that's right, e^x, and it *don't* get much *smoooooother* than me!!"

The operator calmly retorts, "Hey… I'm $\partial/\partial y$."

Problem 2.2.6. *Compute the equation of the plane tangent to the graphs of (a)* $f(x,y) = xy$ *at* $(0,0)$ *and (b)* $f(x,y) = e^{xy}$ *at* $(0,1)$.

Solution. We make use of the formula $z = f(x_0,y_0) + \nabla f(x_0,y_0)\cdot(x-x_0, y-y_0)$ for the tangent plane.

a. $\nabla f(x,y) = (y,x)$ so evaluated at $(0,0)$ gives $(0,0)$. Similarly, $f(0,0) = 0$ so the tangent plane is the xy-plane, $z = 0$.

b. $\nabla f(x,y) = (ye^{xy}, xe^{xy}) = (1,0)$ at $(0,1)$ and $f(0,1) = 1$. Therefore the equation of the tangent plane is $z = 1 + x$.

□

2.3 Section 4, Problems 1, 3

(To be filled in later)