Homework 3 Solution

MATH 20E
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2 Chapter 2

2.3 Section 3, Problems 5, 13(ab), 16

Problem 2.3.5. *Find the equation of the plane tangent to the surface* $z = x^2 + y^3$ *at* $(3, 1, 10)$.

Solution. Note that once the gradient vector is introduced, the equation of the tangent plane looks almost exactly like the equation of the tangent line to a one-variable function f at x_0:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

The 2-dimensional analog, the equation of the tangent plane is simply the above with $\mathbf{x} = (x, y)$, $\mathbf{x}_0 = (x_0, y_0)$ replacing x and x_0, ∇f replacing f' and the dot product replacing ordinary multiplication:

$$z = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

This is a good way to remember it (also it works in 3, 4, 5,... dimensions). Anyway to solve the problem: we have $\nabla f(x, y) = (2x, 3y^2)$ so that $\nabla f(3, 1) = (6, 3)$. $f(3, 1) = 10$ from the fact the given point was $(3, 1, 10)$ which is $(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$. Therefore we have

$$z = 10 + (6, 3) \cdot (x - 3, y - 1) = 10 + 6(x - 3) + 3(y - 1) = 6x + 3y - 11$$

□

Problem 2.3.13.

1. *(To be filled in later)*
2.

Problem 2.3.16. *Calculate* $\nabla h(1, 1, 1)$ *if* $h(x, y, z) = (x + z)e^{x-y}$

Solution. Gradients extend to any number of variables as you want—∇ in 3 dimensions is exactly what you think it is: $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$. So we have the x-component

$$\frac{\partial}{\partial x}\left((x + z)e^{x-y}\right) = (x + z)e^{x-y} + e^{x-y} = (x + z + 1)e^{x-y},$$

using the tried and true product rule (recall y and z stay constant. Refer to Joke 2.2.1 if necessary). Next,

$$\frac{\partial}{\partial y}\left((x + z)e^{x-y}\right) = -(x + z)e^{x-y},$$

no product rule required this time as the outside multiplier does not involve y, and finally

$$\frac{\partial}{\partial z}\left((x + z)e^{x-y}\right) = e^{x-y},$$

as xe^{x-y} (when distributing out the multiplication) is constant in this case (again recall Joke 2.2.1). Therefore

$$\nabla h(x, y, z) = \left((x + z + 1)e^{x-y}, -(x + z)e^{x-y}, e^{x-y}\right),$$

which, evaluated at $(1, 1, 1)$ gives $(3, -2, 1)$. □

2.4 Section 4, Problems 5, 7, 9, 12

Note that in this section, the terms "tangent vector" and "velocity vector" mean the same thing: the first derivative of the vector-valued function. "Velocity vector" strictly speaking isn't a correct term in all situations—it refers specifically to the physical interpretation of a particle tracing out the parameterized path with time. Nevertheless it is quite a useful way of thinking of it.

Problem 2.4.5. *Find the velocity vector of the path* $6t\mathbf{i} + 3t^2\mathbf{j} + t^3\mathbf{k}$.

Solution. The velocity vector is, via straightforward componentwise differentiation, $(6, 6t, 3t^2)$. □

Problem 2.4.7. *The same as above for* $\mathbf{r}(t) = (\cos^2(t), 3t - t^3, t)$. *The vector is* $\mathbf{r}'(t) = (-2\cos t \sin t, 3 - 3t^2, 1)$ *or* $(-\sin(2t), 3 - 3t^2, 1)$ *if you're into trigonometric identities.*

Problem 2.4.9. $\mathbf{c}(t) = (e^t, \cos(t))$. *Same deal.*

Solution. $\mathbf{c}'(t) = (e^t, -\sin(t))$ □

Problem 2.4.13. $\mathbf{c}'(t)$ *for* $\mathbf{c}(t) = (t^2, e^2)$. *Notice how that second term is a constant?* $\mathbf{c}'(t) = (2t, 0)$.

2.5 Section 5, Problems 4, 5(b), 8, 13

Problem 2.5.4. *Verify the chain rule for* $\partial h/\partial x$ *where* $h(x, y) = f(u(x, y), v(x, y))$ *and*

$$f(u, v) = \frac{u^2 + v^2}{u^2 - v^2}, \qquad u(x, y) = e^{-x-y}, \qquad v(x, y) = e^{xy}$$

Solution. You have got to be kidding me. Special request for Ken: don't grade this problem. The chain rule says:

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x}$$

There are a number of reasons why I think this notation is evil. First of all, it obscures what functions are given and what they're varying with respect to. Is h a function of x and y or u and v? Which one is it? Well this is a relic from the days that functions were treated as second-class citizens and variables were the #1. The modern viewpoint is that the name of the variable doesn't mean anything except as a label to distinguish different slots—it is a placeholder. It is more useful to think of h as a genuine function from $\mathbb{R}^2 \to \mathbb{R}$ (ok, ok, really from $\mathbb{R}^2 \setminus (\Delta \cup -\Delta)) \to \mathbb{R}$ where Δ is the diagonal line and $-\Delta$ is the other diagonal sloping the other way, but that's being nitpicky), acting on two guys (u, v) by taking it to the awful quotient above. Worse, there is another conveniently hidden function without a name—it's time to recognize it as a real citizen of the world with all the rights guaranteed to functions: let's call it $\mathbf{F}$. We actually have

$$\mathbf{F}(x, y) = (e^{-x-y}, e^{xy})$$

We conceptualize h as only being able to do what it says it does, so actually $h(x, y) = \frac{x^2+y^2}{x^2-y^2}$ and not all that e to the junk business. To get h to do what the book claims it's doing, we actually have to look at the function $h \circ \mathbf{F}$. Indeed, $h \circ \mathbf{F}(x, y) = h(e^{-x-y}, e^{xy})$ which is exactly what we seek. But in old times it was permissible to omit the $\mathbf{F}$ and so h "varying with respect to x and y" was understood to mean that $\mathbf{F}$ was being applied. The Commission on Functional Rights has lobbied for a number of years to end this senseless discrimination against "implicit transformations." This confused me so much for a number of *years* which was why I shied away from classes in ODEs and PDEs. Now let's write it down the *right* way:

$$\frac{\partial(h \circ \mathbf{F})}{\partial x} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x}$$

There is the method of drawing a tree. Draw the first function, then the first slot, second slot, and finally the x and y. The derivative of the composition with respect to the variable at the bottom is equal to the sum of all possible paths to the bottom; each term being the product of the links in the chain (each link in the chain is the partial derivative of the upper variable with respect to the lower). Ok enough ranting about mathematical experience. We have

$$(h \circ \mathbf{F})(x, y) = \frac{e^{-2(x+y)} + e^{2xy}}{e^{-2(x+y)} - e^{2xy}}$$

Differentiating this with respect to x we get, via the Quotient Rule (courtesy of *Mathematica*):

$$\frac{2\, e^{2\, x\, y}\, y - 2\, e^{-2\, x-2\, y}}{e^{-2\, x-2\, y} - e^{2\, x\, y}} - \frac{\left(e^{-2\, x-2\, y} + e^{2\, x\, y}\right)\left(-2\, e^{2\, x\, y}\, y - 2\, e^{-2\, x-2\, y}\right)}{\left(e^{-2\, x-2\, y} - e^{2\, x\, y}\right)^2}$$

Which `Simplify[]`s to

$$\frac{4\, e^{2\,(y\, x+x+y)}\,(y+1)}{\left(-1 + e^{2\,(y\, x+x+y)}\right)^2}$$

On the other hand, $\nabla h(u, v) =$

$$\left\{\frac{2\, u}{u^2 - v^2} - \frac{2\, u\,(u^2+v^2)}{(u^2-v^2)^2},\ \frac{2\,(u^2+v^2)\, v}{(u^2-v^2)^2} + \frac{2\, v}{u^2 - v^2}\right\}$$

Which simplifies to

$$\left\{-\frac{4\, u\, v^2}{(u^2-v^2)^2},\ \frac{4\, u^2\, v}{(u^2-v^2)^2}\right\}$$

.

Now $\partial\mathbf{F}/\partial x = (-e^{-x-y}, ye^{xy})$ and so $\nabla h \cdot \partial\mathbf{F}/\partial x =$

$$\frac{4\, e^{2\, y\, x-2\, x-2\, y}\, y}{\left(e^{-2\, x-2\, y} - e^{2\, x\, y}\right)^2} + \frac{4\, e^{2\, y\, x-2\, x-2\, y}}{\left(e^{-2\, x-2\, y} - e^{2\, x\, y}\right)^2}$$

. This reduces via `Simplify[]` to

$$\frac{4\, e^{2\,(y\, x+x+y)}\,(y+1)}{\left(-1 + e^{2\,(y\, x+x+y)}\right)^2}$$

agreeing with our previous result. □

Problem 2.5.5. *Verify the chain rule for* $f(x, y) = e^{xy}$, $\mathbf{c}(t) = (3t^2, t^3)$. *Here they actually say* $f \circ \mathbf{c}$.

Solution. First, via straight substitution, $(f \circ \mathbf{c})(t) = e^{3t^2 t^3} = e^{3t^5}$. Therefore $(f \circ \mathbf{c})'(t) = 3 \cdot 5t^4 e^{3t^5} = 15t^4 e^{t^5}$. Much more manageable than the last. Using the chain rule,

$$\frac{d(f \circ \mathbf{c})}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

which is, after substituting

$$ye^{xy}6t + xe^{xy}3t^2 = t^3 e^{3t^5}6t + 3t^2 e^{3t^5}3t^2 = 6t^4 e^{3t^5} + 9t^4 e^{3t^5} = 15t^4 e^{3t^5}$$

agreeing with the previous. □

Problem 2.5.8. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Making the substitution*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \sin\varphi \sin\theta \\ \rho \sin\varphi \sin\theta \\ \rho\cos\varphi \end{pmatrix}$$

(the spherical coordinate transformation) into $f(x, y, z)$ compute the partial derivatives of the composed function with respect to ρ, φ, and θ in terms of the partial derivatives of f.

Solution. Aaargh! They do it again, writing $\partial f/\partial\theta$ instead of a more civilized $\partial(f \circ \mathbf{F})/\partial\theta$, where $\mathbf{F}$ is the official transformation map:

$$\mathbf{F}\begin{pmatrix} \rho \\ \varphi \\ \theta \end{pmatrix} = \begin{pmatrix} \rho \sin\varphi \sin\theta \\ \rho \sin\varphi \sin\theta \\ \rho\cos\varphi \end{pmatrix}$$

We use the chain rule to compute $\mathbf{D}(f \circ \mathbf{F})$:

$$\mathbf{D}(f \circ \mathbf{F}) = [\mathbf{D}f][\mathbf{DF}] = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial(f\circ\mathbf{F})}{\partial\rho} & \frac{\partial(f\circ\mathbf{F})}{\partial\varphi} & \frac{\partial(f\circ\mathbf{F})}{\partial\theta} \end{bmatrix}$$

Plugging in **DF** we get

$$\begin{bmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{bmatrix}$$

so

$$\begin{aligned} \frac{\partial(f\circ\mathbf{F})}{\partial\rho} &= \frac{\partial f}{\partial x}\sin\varphi\cos\theta + \frac{\partial f}{\partial y}\sin\varphi\sin\theta + \frac{\partial f}{\partial z}\cos\varphi, \\ \frac{\partial(f\circ\mathbf{F})}{\partial\varphi} &= \frac{\partial f}{\partial x}\rho\cos\varphi\cos\theta + \frac{\partial f}{\partial y}\rho\cos\varphi\sin\theta - \frac{\partial f}{\partial z}\rho\sin\varphi, \text{ and} \\ \frac{\partial(f\circ\mathbf{F})}{\partial\theta} &= -\frac{\partial f}{\partial x}\rho\sin\varphi\sin\theta + \frac{\partial f}{\partial y}\rho\sin\varphi\cos\theta \end{aligned}$$

□

Problem 2.5.13. *A duck is swimming... (to be filled in later).*

2.6 Section 6: Problem 1, 2b, 3a, 4ac, 6, 15

Ahh, the Gradient Vector. Good ol' ∇ (called the **del** or **nabla** operator (*nabla* comes from the word for a particular stringed instrument which is in the shape of ∇)). This operator is one of the stars of this course. It is best conceptualized as stacks of level-surface—the closer together level surfaces are, the faster the function is changing, so the greater the magnitude of the gradient. The direction of the gradient is perpendicular to these surfaces, because it points in the direction of *increase* of f, while, moving along the level surface doesn't change the value (hence the name *level*).

Directional derivatives come into play when something such as a tangent vector is threading *across* these surfaces—it gives a measure of how many surfaces you're intersecting at a particular time, i.e. a rate of change in surfaces traversed (increase in f). In fact this property leads many a mathematician to call the gradient vector a *cotangent* vector (well you hear the term tangent employed, why not cotangent?) or sometimes tangent *covector*. Geeky, I know. Cotangent vectors are very important in mathematics; however, a full description of it would take us on a (co)tangent and unduly lengthen this solution.

Problem 2.6.1. *Show the directional derivative of* $f(x, y, z) = z^2x + y^3$ *at* $(1, 1, 2)$ *in the direction* $\mathbf{u} = (1/\sqrt{5}, 2/\sqrt{5}, 0)$ *is* $2\sqrt{5}$.

Solution. This is just definition-pushing: compute $\nabla f \cdot \mathbf{u}$. $\nabla f = (z^2, 3y^2, 2zx)$ so at $(1, 1, 2)$ we get $\nabla f = (4, 3, 4)$. This dotted with $(1/\sqrt{5}, 2/\sqrt{5}, 0)$ gives $4/\sqrt{5} + 6/\sqrt{5} = 10/\sqrt{5} = 10\sqrt{5}/5 = 2\sqrt{5}$ as promised. □

Problem 2.6.2. *Compute the direction deriv. of* $f(x, y) = \log\sqrt{x^2 + y^2}$ *at* $(1, 0)$ *in the direction* $(2/\sqrt{5}, 1/\sqrt{5})$.

Solution. $f_x(x, y) = \frac{1}{2}\frac{\partial}{\partial x}(\log(x^2 + y^2)) = \frac{1}{2}(2x/(x^2 + y^2)) = x/(x^2 + y^2)$. The observant reader may have noted that we have used the same trick as something on homework solution 2 namely taking out a fraction of $1/2$ in the log to assist us. By the symmetry of x and y, which the observant reader may have noted which helped us again in the very same problem, leads us to find $f_y(x, y) = y/(x^2+y^2)$. The unobservant reader need not worry, for yours truly has not been the best of readers at times. Therefore we have

$$\nabla f(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right),$$

evaluating at $(1, 0)$ we get $(1, 0)$ (a lot of effort for not changing the result!). This dotted with the direction vector gives the first component, and so the result is $2/\sqrt{5} = \frac{2}{5}\sqrt{5}$ (if you're into rationalizing denominators. I'm not. Hint: unless it's really obviously going to save you something, I won't care if you leave an unrationalized denominator on an answer to a test question =) □

Problem 2.6.3. *Compute the dir. deriv along unit vectors parallel to the given:* $f(x, y) = x^y$ *at* (e, e) *with* $\mathbf{d} = (5, 12)$.

Solution. Note that $5^2 + 12^2 = 13^2$ (another Pythagorean triple worth remembering [I remember 3, 4, 5; 5, 12, 13; and 7, 24, 25]) so that one vector you want to dot with is $\mathbf{u} = (5/13, 12/13)$. But there's another unit vector parallel to it, namely $-\mathbf{u}$ which will give exactly minus the answer (derivative is a linear operation, remember) so it suffices to check the first case out.

The partial derivatives of x^y are an interesting story in itself, because it tells the story of the raising to powers operation. For operations in the base, namely taking $(\partial/\partial x)(x^y)$ we *hold y constant*—cf. Joke 2.2.1 so this is nothing but the venerable power rule yx^{y-1}. On the other hand operation in the exponent $\partial/\partial y$ is slightly harder, as it gets into exponentials. If you remember $x^y = e^{y\ln x}$ then the derivative is easier to see: $(\partial/\partial y)x^y = (\ln y)e^{x\ln y} = (\ln y)x^y$. Therefore $\nabla f(x,y) = (yx^{y-1}, (\ln y)x^y)$. Plugging in (e,e) we have $\nabla f(e,e) = (ee^{e-1}, (\ln e)e^e) = (e^e, e^e)$. Lots of e's. Believe it or not, e^e is a somewhat useful constant in math as well. It tells many stories of the x^y function of 2 variables. Once again we have gone on a cotangent. There doesn't seem to be anything special about $\nabla f(e,e) \cdot (5/13, 12/13) = 17e^e/13$. □

Problem 2.6.4. *Find planes tangent to*

a. $x^2 + 2y^2 + 3xz = 10$ *at* $(1, 2, \frac{1}{3})$

c. $xyz = 1$ *at* $(1, 1, 1)$

Solution. Recall that any vector determines a plane which is perpendicular to it, given a base point. So given a gradient vector, and a point, it defines a plane normal to it. But we also know that the gradient vector is normal to the level surface on which the point in question sits—remember the introduction to this section. So if both a plane and a surface are normal to a vector at a certain point, they must be tangent. So to define the tangent plane we take the given point as $\mathbf{r}_0$, $\mathbf{r} = (x,y,z)$, and the equation of the tangent plane is given by the equation $\nabla f(\mathbf{r}_0) \cdot (\mathbf{r} - \mathbf{r}_0) = 0$.

a. This is the level surface with value 10 of the function $f(x,y,z) = x^2 + 2y^2 + 3xz$. Therefore $\nabla f(x,y,z) = (2x + 3z, 4y, 3x) = (3, 8, 3)$. So the equation is $(3,8,3) \cdot ((x,y,z) - (1, 2, \frac{1}{3}) = 0$ or $3(x-1) + 8(y-2) + 3(z - \frac{1}{3}) = 0$, which further reduces to $3x + 8y + 3z = 20$.

c. Taking $f(x,y,z) = xyz$ this is the level surface $f(x,y,z) = 1$. $\nabla f(x,y,z) = (yz, xz, xy)$, which gives $(1,1,1)$ at $(1,1,1)$. So much trouble for no change. So the equation of the tangent plane is $(1,1,1) \cdot ((x,y,z) - (1,1,1)) = 0$ or $x - 1 + y - 1 + z - 1 = 0$, reducing to $x + y + z = 3$.

□

Problem 2.6.5. *Compute* ∇f *for each of the following:*

a. $f(x,y,z) = 1/\sqrt{x^2 + y^2 + z^2}$

b. $f(x,y,z) = xy + yz + xz$

c. $f(x,y,z) = \frac{1}{x^2+y^2+z^2}$

Solution. These guys really like symmetry or something—namely, if you somehow permute the variables like swap x for y you get exactly the same expression. So in each case it suffices to compute $\partial f/\partial x$ and infer the others by permuting the variables.

a. $\partial f/\partial x = (-1/2)(x^2 + y^2 + z^2)^{-3/2} 2x = -x/(x^2 + y^2 + z^2)^{3/2}$. Therefore by argument from symmetry

$$\nabla f(x,y,z) = \left(-\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}, -\frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right)$$

b. $\partial f/\partial x = y + z$ so by symmetry

$$\nabla f(x,y,z) = (y + z, x + z, y + x)$$

c. $\partial f/\partial x = (-1/(x^2+y^2+z^2)^2)2x = -2x/(x^2+y^2+z^2)^2$. So

$$\nabla f(x,y,z) = \left(-\frac{2x}{(x^2+y^2+z^2)^2}, -\frac{2y}{(x^2+y^2+z^2)^2}, -\frac{2z}{(x^2+y^2+z^2)^2}\right)$$

□

Problem 2.6.15. *Let* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and* $r = \|\mathbf{r}\|$. *Then*

$$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$$

Solution. Note that $r = \sqrt{x^2+y^2+z^2}$. So $1/r = 1/\sqrt{x^2+y^2+z^2}$. Do you get a feeling of déjà vu? You see number 6a? We've done this already, namely

$$\nabla\left(\frac{1}{r}\right) = \left(-\frac{x}{(x^2+y^2+z^2)^{3/2}}, -\frac{y}{(x^2+y^2+z^2)^{3/2}}, -\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

Notice all the denominators are the same: $(x^2+y^2+z^2)^{3/2} = \sqrt{x^2+y^2+z^2}^3 = r^3$. So

$$\nabla\left(\frac{1}{r}\right) = \frac{1}{r^3}(-x,-y,-z) = \frac{-\mathbf{r}}{r^3}$$

as promised. □

3 Chapter 3

3.2 Section 2, Problems 2, 5

Problem 3.2.1. *Determine the second-order Taylor Formula for* $f(x,y) = 1/(x^2+y^2+1)$ *at* $(x_0,y_0) = (0,0)$.

Solution. We write the solution in the form

$$P_2(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0)(x-x_0,y-y_0) + \frac{1}{2}(x-x_0,y-y_0)[Hf]\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix},$$

where $[Hf] = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$ which can be shown to be exactly the same as that horrid looking double sum with the h_ih_j's. $f(0,0) = 1$, $\nabla f(x,y) = \left(-2x/(x^2+y^2+1)^2, -2y/(x^2+y^2+1)^2\right)$ The result should be that only the diagonal terms are nonzero, and give the value of -2 for each. Those will cancel with the 1/2. Therefore the 2nd order poly is $P_2(x,y) = 1 - x^2 - y^2$. More details forthcoming. □

More to come