Homework 4 Solution

MATH 20E

CHRIS TIEE

4 Chapter 4

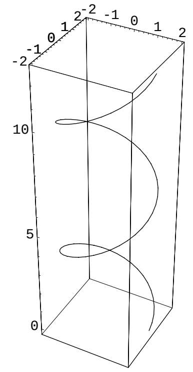
4.2 Section 2, Problems 1, 3, 8, 12.

Arc length is cool but notoriously hard to compute.

Problem 4.2.1. *Calculate the length of* $\mathbf{c}(t) = (2\cos t, 2\sin t, t)$ *for* $0 \le t \le 2\pi$.

Solution. This curve is a helix, a relatively "standard" curve to add to your stock. $\mathbf{c}'(t) = (-2\sin t, 2\cos t, 1)$ so that $\|\mathbf{c}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$. Therefore

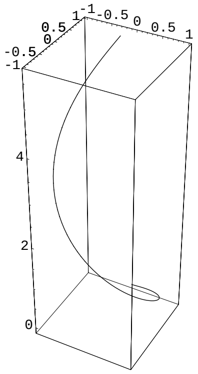
$$\int_0^{2\pi} \|\mathbf{c}'(t)\|\, dt = \int_0^{2\pi} \sqrt{5}\, dt = 2\pi\sqrt{5}.$$



□

Problem 4.2.3. *Let* $\mathbf{c}(t) = (\sin 3t, \cos 3t, 2t^{3/2})$. *Find the length of this path from* $t = 0$ *to* 1.

Solution. This may be called the "semicubical helix" due to the $t^{3/2}$ in the last term. In any case it looks like a helix except the distance between the coils gets larger and larger:



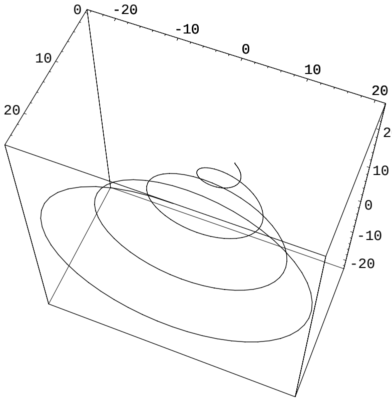
$\mathbf{c}'(t) = (3\cos 3t, -3\sin 3t, 3t^{1/2})$ so that $\|\mathbf{c}'(t)\| = \sqrt{9\cos^2 3t + 9\sin^2 3t + 9t} = \sqrt{9+9t} = 3\sqrt{1+t}$. Integrating this guy from 0 to 1 we have

$$\int_0^1 3\sqrt{1+t}\,dt = \int_1^2 3\sqrt{u}\,du = 2u^{3/2}\Big|_1^2 = 2(2^{3/2}-1) = 2(2\sqrt{2}-1).$$

□

Problem 4.2.8. *Let* $\mathbf{c}(t)$ *be the path* $\mathbf{c}(t) = (t, t\sin t, t\cos t)$*. Find the arc length of* $\mathbf{c}$ *between the two points* $(0,0,0)$ *and* $(\pi, 0, -\pi)$*.*

Solution. The path is a conical spiral:



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To find the values of t so we can integrate, we simply solve the three equations for t at each point:

$$t = 0, \quad t\sin t = 0, \quad t\cos t = 0$$

But since the first component gives us t directly, so the start point is $t = 0$. Similarly solving at $(\pi, 0, -\pi)$ gives us $t = \pi$ as the end point. In general when solving equations of this kind, all three equations have to be compatible; if you get more than one solution for some equations, use the most restrictive combination (e.g. if you get $t = 1$, $t = 2$ for one equation and $t = 1$, $t = -1$ from another, only $t = 1$ will work). Taking derivatives we have

$$\mathbf{c}'(t) = (1, t\cos t + \sin t, -t\sin t + \cos t)$$

and taking its length we have

$$\|\mathbf{c}'(t)\| = \sqrt{1 + (t^2\cos^2 t + 2t\cos t\sin t + \sin^2 t) + (t^2\sin^2 t - 2t\sin t\cos t + \cos^2 t)}$$

which, fortunately, results in cancellations. Using $\sin^2 t + \cos^2 t = 1$ twice, we get

$$\|\mathbf{c}'(t)\| = \sqrt{2+t^2}$$

so meaning that we must evaluate the following:

$$\int_0^\pi \sqrt{2+t^2}\,dt$$

which does not look pleasant to evaluate. In fact arc length integrals are notoriously difficult to compute without some sort of tricky cancellation or getting a perfect square under that square root. Taking $t = \sqrt{2}\sinh u$ which, I'm sure you all remember with crystal clear accuracy from 20B (just kidding =) is a hyperbolic substitution, we find $dt = \sqrt{2}\cosh u\ du$. Setting the limits, $t = 0$ means $u = 0$ (since $\sinh(0) = 0$) and the other limit is the rather unpleasant $u = \sinh^{-1}(\pi/\sqrt{2})$. Noting that $1 + \sinh^2 u = \cosh^2 u$ we substitute this into the integral:

$$\int_0^{\sinh^{-1}(\pi/\sqrt{2})} \sqrt{2\cosh^2 u}\,\sqrt{2}\cosh u\ du = \int_0^{\sinh^{-1}(\pi/\sqrt{2})} 2\cosh^2 u\ du$$

Now we use the definition of $\cosh u = \frac{e^u+e^{-u}}{2}$ so we've got

$$\int_0^{\sinh^{-1}(\pi/\sqrt{2})} \frac{e^{2u} + 2 + e^{-2u}}{2}\ du = \int_0^{\sinh^{-1}(\pi/\sqrt{2})} (1 + \cosh(2u))\ du$$

(using the definition again… I wasn't sure which "trig" double angle identity to use for $\cosh^2$, hehe). And evaluating, we have

$$\left[u + \frac{1}{2}\sinh(2u)\right]_0^{\sinh^{-1}(\pi/\sqrt{2})} = \sinh^{-1}\left(\frac{\pi}{\sqrt{2}}\right) + \frac{\pi}{\sqrt{2}}\sqrt{\frac{\pi^2}{2} + 1}$$

which, after 30 microseconds of mental calculation, comes out to be about 6.95025078632747008278… (mental calculation… for my computer =). □

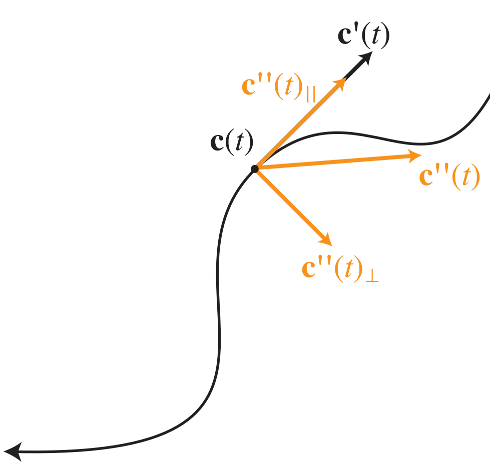
Problem 4.2.12. *Let* $\mathbf{c} : [a,b] \to \mathbb{R}^3$ *be an infinitely differentiable path (called* C^∞ *by most math geeks). Assume* $\mathbf{c}'(t) \neq \mathbf{0}$ *for any* t*. The vector* $\mathbf{c}'(t) = \|\mathbf{c}'(t)\| =: \mathbf{T}(t)$ *is tangent to* $\mathbf{c}$ *at* $\mathbf{c}(t)$ *and, because* $\|\mathbf{T}(t)\| = 1$ *for all* t, $\mathbf{T}$ *is called the* unit tangent vector *to* $\mathbf{c}$.

a. *Show that* $\mathbf{T}'(t)\cdot\mathbf{T}(t) = 0$ Hint*: differentiate* $\mathbf{T}(t)\cdot\mathbf{T}(t) = 1$.

b. *Write down a formula for* $\mathbf{T}(t)$ *in terms of* $\mathbf{c}$.

Solution.

a. The solution to this is a very cute trick. The hint is used in many places in differential geometry. The trick is in realizing $\frac{d}{dt}(\mathbf{T}(t)\cdot\mathbf{T}(t)) = 0$ because $\mathbf{T}(t)\cdot\mathbf{T}(t) = \|\mathbf{T}(t)\|^2 = 1$, a constant. But using the product rule, which holds for dot products as well, we have $\mathbf{T}'(t)\cdot\mathbf{T}(t) + \mathbf{T}(t)\cdot\mathbf{T}'(t) = 0$. But the two terms in that sum are the same, hence $2\mathbf{T}'(t)\cdot\mathbf{T}(t) = 0$, which gives us the result. Despite the trickery involved in computing this, it really means something: when a vector changes, the two things that make up a vector, the magnitude and the direction change. The magnitude of vector, it turns out, changes depending only on the parallel (tangential) component of its derivative, while the direction changes depending on the perpendicular (normal) component (see diagram below showing this for the vector $\mathbf{c}'(t)$; here we have shown $\mathbf{c}''(t)$ in orange broken up into the two components $\mathbf{c}''(t)_{\|}$ and $\mathbf{c}''(t)_{\perp}$. And since the magnitude of $\mathbf{T}$ does not change at all, the parallel component is 0 and so $\mathbf{T}' \perp \mathbf{T}$.



b. We simply compute $\frac{d}{dt}(\mathbf{c}'(t)/||\mathbf{c}'(t)||)$ which just uses the usual quotient rule:

$$\frac{d}{dt}\mathbf{T}(t) = \frac{d}{dt}\frac{\mathbf{c}'(t)}{||\mathbf{c}'(t)||} = \frac{||\mathbf{c}'(t)||\mathbf{c}''(t) - \mathbf{c}'(t)||\mathbf{c}'(t)||'}{||\mathbf{c}'(t)||^2}$$

Leaving it in this form is acceptable, though we can express $||\mathbf{c}'(t)||'$ by using $||\mathbf{c}'(t)|| = \sqrt{\mathbf{c}' \cdot \mathbf{c}'}$ so that

$$||\mathbf{c}'(t)||' = \frac{1}{2\sqrt{\mathbf{c}'(t) \cdot \mathbf{c}'(t)}} 2\mathbf{c}''(t) \cdot \mathbf{c}'(t) = \mathbf{c}''(t) \cdot \frac{\mathbf{c}'(t)}{||\mathbf{c}'(t)||}$$

(this proves, by the way, the statement in part a) that the derivative of the magnitude of a vector only depends on the component parallel to it). Plugging this into the above (and suppressing extra (t)'s for clarity) we have

$$\frac{d\mathbf{T}}{dt} = \frac{\mathbf{c}''}{||\mathbf{c}'||} - \frac{(\mathbf{c}'' \cdot \mathbf{c}')\mathbf{c}'}{||\mathbf{c}'||^3}$$

which is quite unpleasant. You may freely skip to the next problem now, but interested readers may like to see the derivation of a more elegant result from the above. An old professor once told me, many things can be simplified by adding zero or multiplying by 1. We can take $1 = \mathbf{T} \cdot \mathbf{T}$ (just like in the last problem) and let:

$$\frac{d\mathbf{T}}{dt} = \frac{\mathbf{c}''(\mathbf{T} \cdot \mathbf{T})}{||\mathbf{c}'||} - \frac{(\mathbf{c}'' \cdot \mathbf{c}')\mathbf{c}'}{||\mathbf{c}'||^3} = \frac{\mathbf{c}''}{||\mathbf{c}'||}\left(\frac{\mathbf{c}'}{||\mathbf{c}'||} \cdot \frac{\mathbf{c}'}{||\mathbf{c}'||}\right) - \frac{(\mathbf{c}'' \cdot \mathbf{c}')\mathbf{c}'}{||\mathbf{c}'||^3} = \frac{\mathbf{c}''(\mathbf{c}' \cdot \mathbf{c}') - \mathbf{c}'(\mathbf{c}' \cdot \mathbf{c}')}{||\mathbf{c}'||^3}$$

Now for the famous BAC-CAB vector identity[1] which gives us

$$\frac{d\mathbf{T}}{dt} = \frac{\mathbf{c}' \times (\mathbf{c}'' \times \mathbf{c}')}{||\mathbf{c}'||^3} = \mathbf{T} \times \frac{\mathbf{c}'' \times \mathbf{c}'}{\mathbf{c}' \cdot \mathbf{c}'}$$

Note that by the geometric property of the cross product, namely that it is orthogonal to the vectors which compose the product, we get an alternate (more meaningful) proof that $\mathbf{T} \perp \mathbf{T}'$.

□

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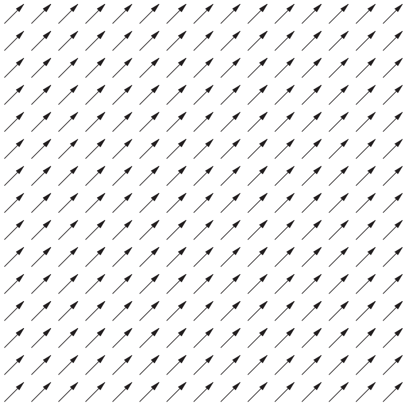
[1] $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$

4.3 Section 3, Problems 1, 3, 7, 8, 13, 20

Vector fields are cool but notoriously difficult to draw. The brute-force way to do it, given the field $\mathbf{F}$, is to sketch at a bunch of grid points a vector which is the value of $\mathbf{F}$. The problem with this approach is that sometimes vectors get long so things overlap and you can't see anything useful. So the thing to do is to scale down the vector field i.e. to actually sketch some small multiple of the field in question. If the vector field is the gradient of some function (e.g. you are asked to draw ∇f for some scalar function f) then your job is often made *a lot* easier because you can sketch the level sets (curves or surfaces, depending on the dimension); the gradient, if you recall, is perpendicular to them[2].

Problem 4.3.1. *Sketch the field* $\mathbf{F}(x, y) = (2, 2)$

Solution. This is a constant vector field, so the vector will be pointing in the same direction at every point. Here's a picture:



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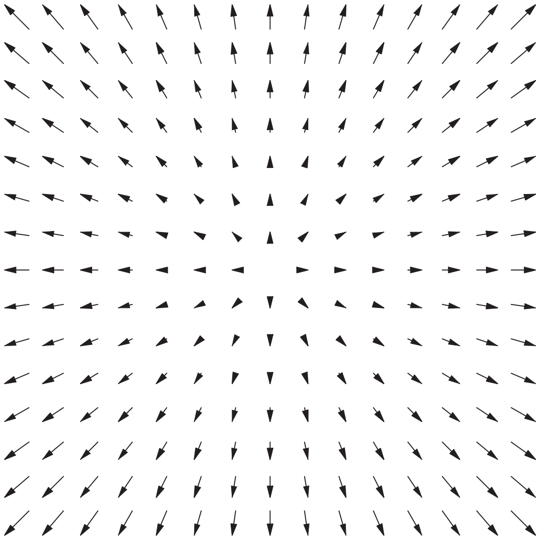
Problem 4.3.3. *Sketch the field* (x, y).

Solution. We could do it by brute force. Here's a picture:



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[2] Physicists will call these things *equipotential contours* or *equipotential surfaces.*

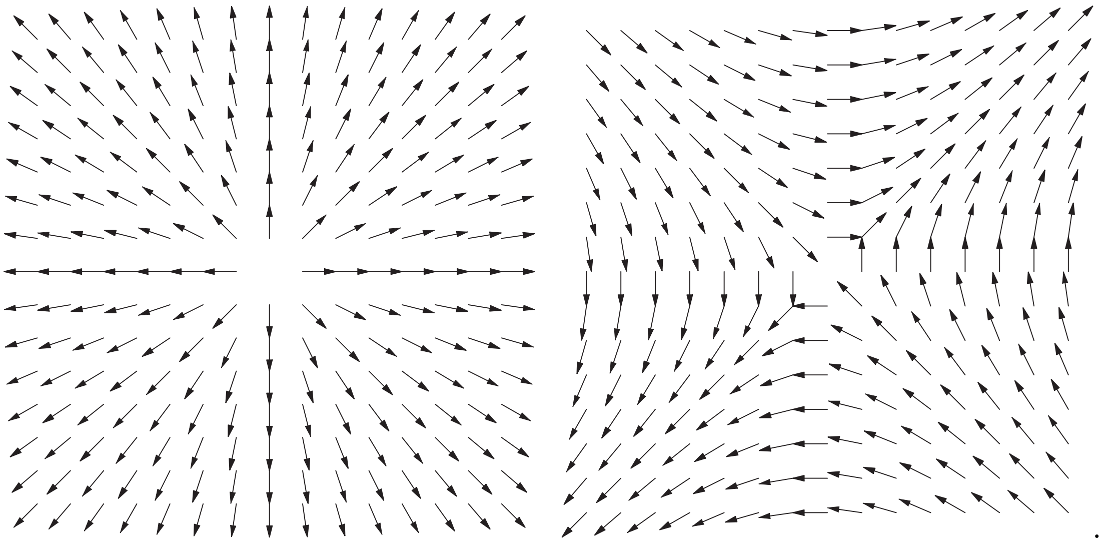
Another way is to realize that this is a gradient vector field (in the next section I'll explain how to make guesses at whether or not it is just by looking). This is a good time to illustrate a good technique: if we are to solve $\nabla f = \mathbf{F} = (F_1, F_2)$ we're solving the two partial differential equations

$$\frac{\partial f}{\partial x} = F_1, \quad \frac{\partial f}{\partial y} = F_2.$$

So one way is to solve the first one and see if we can make it fit to the second one. Obviously this isn't always going to work but it's worth a try. For this one, since the first component is x we know, if there is a f for which $\mathbf{F} = \nabla f$, then $\partial f/\partial x = x$. So by "partial integration" we get $f(x,y) = \frac{1}{2}x^2 + C$. But "partial integration" with respect to x isn't going to yield a genuine constant of integration—only constant with respect to x. Namely the C abovve is actually a function $C(y)$. How do we find $C(y)$? Well, take the partial derivative with respect to y: $\partial f/\partial y = C'(y)$ (the x goes away by recalling Joke 2.2.1). But we know what $\partial f/\partial y$ is; it has to be y since we are supposing $\nabla f = (x,y)$. Therefore $C'(y) = y$ so that $C(y) = \frac{1}{2}y^2 + K$ where K is a genuine constant this time (since we know that it doesn't depend on x). Plugging in gives us $f(x,y) = \frac{1}{2}(x^2 + y^2) + K$ for any constant K, that is to say $\mathbf{F}$ is the gradient of many functions (just like how the indefinite integral is only determined up to a constant). But constants don't affect the shape of the level curves (only their labeling) so, looking at the level curves of this guy, we get a bunch of circles which get closer and closer with larger radii. So the vector field points perpendicular to these circles, namely radially outward, as expected. □

Problem 4.3.7 (7 and 8). *Sketch the fields* $\mathbf{F}(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ *and* $\mathbf{F}(x,y) = \left(\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right)$.

Solution. The formulas two vector fields look deceptively similar. But the fields don't look at all alike:



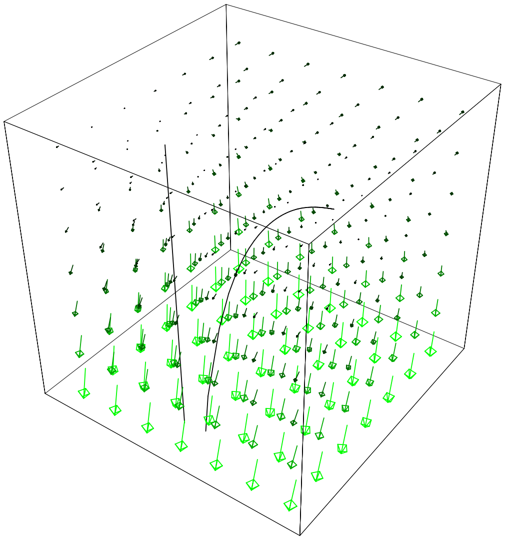
Notice that every vector in both fields is of unit length (that much is visible by noticing the denominators are the lengths of (x,y) and (y,x). So #6 looks like the field in problem 3 except only the magnitudes are different. You can start developing your "vector field recognition skills" right away by noticing anything multiplying (x,y) in it is going to look somewhat radial like the first one above (trivia: called an *index 1* field), while (y,x) corresponds to hyperbolic-shaped fields (*index* -1 fields)—the second one (also useful is $(-y,x)$, a circulating field. I will have much more to say about these in the next section on curl and divergence). □

Problem 4.3.13. *Show that* $\mathbf{c}(t) = (e^{2t}, \log|t|, 1/t)$ *is a flow line of the velocity vector field* $\mathbf{F}(x, y, z) = (2x, z, -z^2)$.

Solution. Flow lines are cool. Yes, I have the annoying habit of saying everything in this class is cool. But it is true. Flow lines are, at least for gradient vector fields, a complementary ("dual") concept to level curves or surfaces.[3] Basically flow lines are exactly what you think they are—if the vector field describes the velocity of a fluid, then the flow lines, tangent to these velocity vectors, describe the actual path taken by the fluid. The key concept is, the tangent vector to a point on a flow line is exactly equal to the value of the vector field at that point, i.e.

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)).$$

The verification of this problem is therefore quite easy: $\mathbf{c}'(t) = (2e^{2t}, 1/t, -1/t^2)$, while $\mathbf{F}(\mathbf{c}(t)) = \mathbf{F}(e^{2t}, \log|t|, 1/t) = (2(e^{2t}), 1/t, -(1/t)^2) = (2e^{2t}, 1/t, -1/t^2)$ which exactly matches the formula for $\mathbf{c}'(t)$. See picture.



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Problem 4.3.20. *Sketch the gradient field* $-\nabla V$ *for* $V(x, y) = (x + y)/(x^2 + y^2)$.

Solution. As I always like to say, if you're given that your function is the gradient of something, *level curves are your friends*. But even so, it is not at all obvious, *a priori*[4] what the heck this thing should look like. Even when I drew this this vector field using those curves as a guide, it still didn't look anything like what the computer showed me.

To calculate the gradient field, it is a fairly gnarly application of the Quotient rule, but symmetry saves the day (as usual) in that we only have to compute $\partial V/\partial x$ and the other partial comes by swapping the variables:

$$-\frac{\partial V}{\partial x} = -\frac{(x^2 + y^2) - (x + y)(2x)}{(x^2 + y^2)^2} = \frac{x^2 - y^2 + 2xy}{(x^2 + y^2)^2}$$

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[3] Physicists, when dealing with flow lines of a force field, call these the "lines of force." This concept was popularized by Michael Faraday, who used this to picture force fields, without any mathematical training. "The Force is an energy field that surrounds everything"—Obi-Wan Kenobi (Sorry, Obi-Wan… the Force is the *gradient* of the energy field surrounding everything, not the energy field itself!

[4] Cheers to my old roommate Michael VanValkenburgh and countless mathematical reference for the use of that phrase…

so that

$$-\nabla V = \left(\frac{x^2 - y^2 + 2xy}{(x^2+y^2)^2}, \frac{y^2 - x^2 + 2xy}{(x^2+y^2)^2}\right) \tag{4.1}$$

As remarked before this is hopeless to try to plot by sampling points. Instead we can get some insight by looking at the level curves $V(x,y) = c$. Then we have $(x+y)/(x^2+y^2) = c$ or $x^2 + y^2 = (x+y)/c$. Sorting everything out we have

$$x^2 - \frac{x}{c} + y^2 - \frac{y}{c} = 0$$

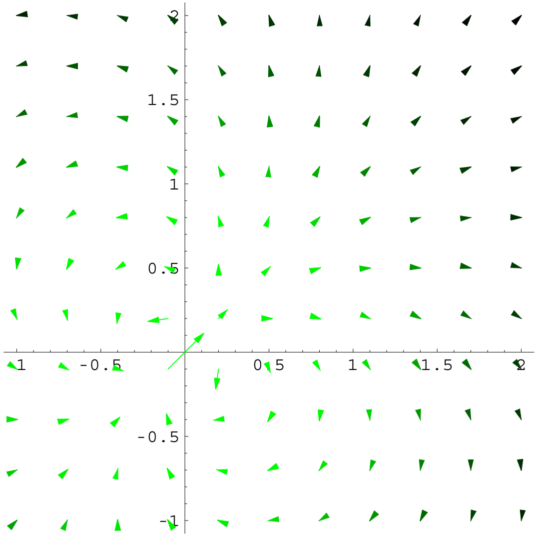
This suggests completing the square, by adding $\frac{1}{4c^2}$ to each term and to both sides:

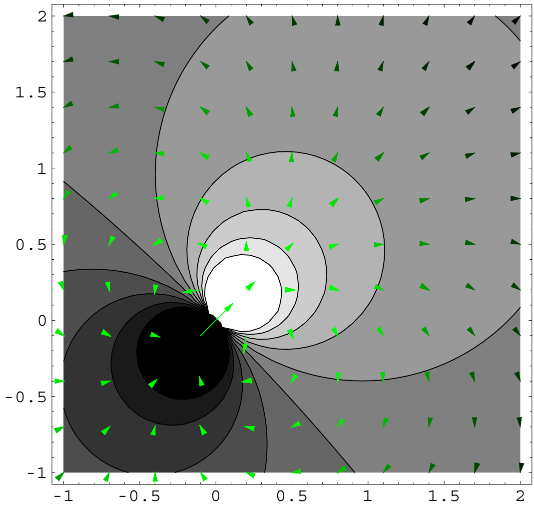
$$x^2 - \frac{x}{c} + \frac{1}{4c^2} + y^2 - \frac{y}{c} + \frac{1}{4c^2} = \frac{1}{2c^2}$$

or finally

$$\left(x - \frac{1}{2c}\right)^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{2c^2}$$

These are circles in the plane (much more informative than (4.1) above). The direction of the gradient is going to be perpendicular and pointing toward *decreasing V* (since we're talking about the *negative* gradient). The magnitudes will correspond to how bunched up the level curves are. Here's a picture of everything:





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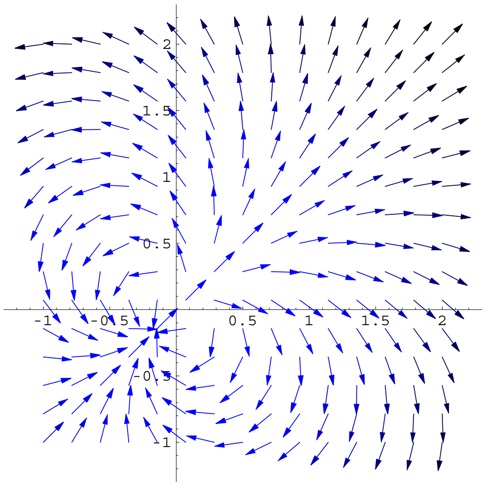
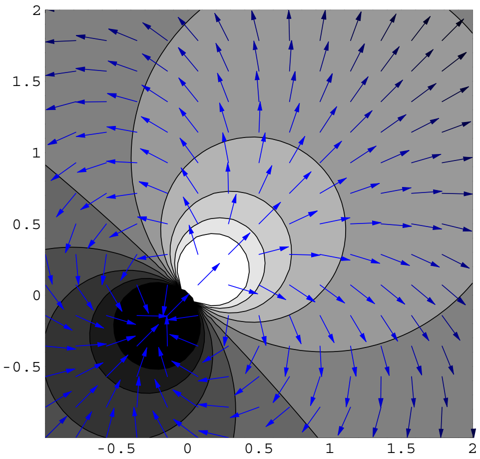
You might notice some of these vectors are missing tails. The problem is, the level curves (with the level curves to the right: higher regions shaded lighter) all bunch up around the origin since the function blows up there. So the gradient, being the max rate of change of the function, gets very large, and so length-scaling causes most of the tails to become so short that the arrowhead obscures it completely. So here's a picture which compensates for magnitudes (instead of the length of the arrow representing the magnitude, the blueness of it does):

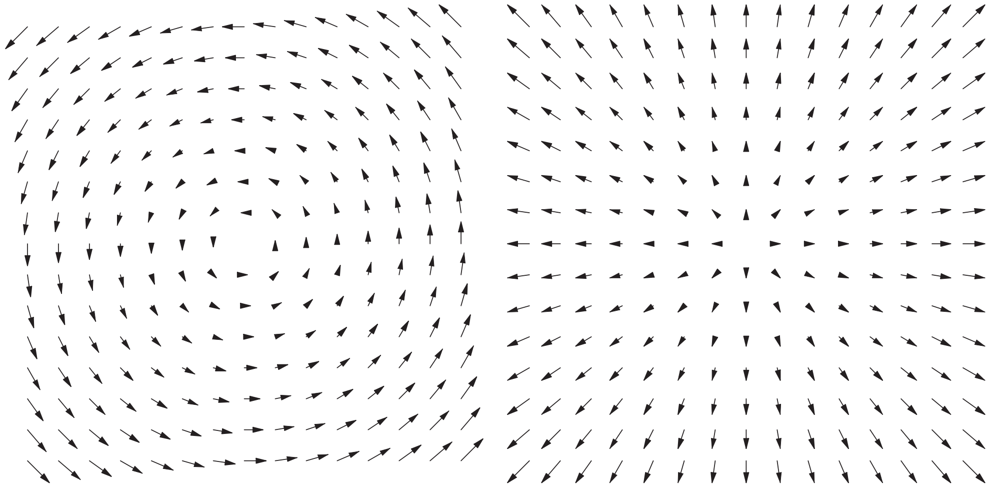




Pretty, huh? Mathematicians say this field is an *index 2* field because of the way it turns around the origin. □

4.4 Section 4, Problems 1, 2, 10, 13, 14, 25, 30

Ahh yes, the heart of the course: differentiating vector fields, and those three famous operations div, grad, and curl[5]. You already know our good friend, the gradient, as measuring max rate of increase of a function, or when looking at a graph, which direction is “uphill.” Curl and divergence measure the tendency, roughly speaking, for vector fields to look respectively like



i.e. the tendency of vector fields to circulate about a point (“curl around” a point hence the name) and to spread out from some source (“diverge from” a source). Also, in terms of flow lines, divergence measures how many lines come out of a surface surrounding a point, and curl measures their tendency to form loops. Major Caveat: there are some evil vector fields which look like the left above and yet still have zero curl—see the supplemental “Gradient vs. Curl” document I’ve put up). Aside from their picturesque meanings and also their importance for applications, we come by a lot of good theoretical results with these guys.

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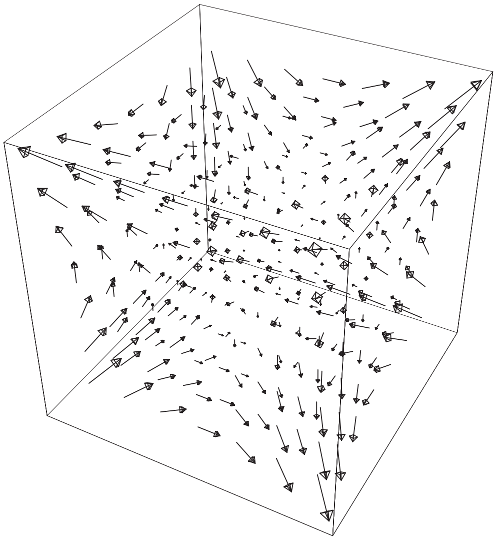
[5] If anyone ever gets around to teaching differential forms to undergrads, one will realize that they are actually all the same operation. Alas, the development of that theory is quite complicated. I refer the interested reader to *Vector Calculus, Linear Algebra, and Differential Forms: a Unified Approach* by John H. Hubbard and Barbara Burke Hubbard)

Problem 4.4.1. *Calculate the divergence of* $\mathbf{V}(x, y, z) = (e^{xy}, -e^{xy}, e^{yz})$

Solution. Divergence is generally a straightforward calculation. $ye^{xy} - xe^{xy} + ye^{yz} = (y - x)e^{xy} + ye^{yz}$. □

Problem 4.4.2. *Calculate the divergence of* $\mathbf{V}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

Solution. Notice that the first component has no x, the second has no y and the third has no z, even though we're supposed to differentiate each one with respect to that variable being omitted! Therefore the divergence is 0 (see picture below—we spice things up by including a 3D vector field!).



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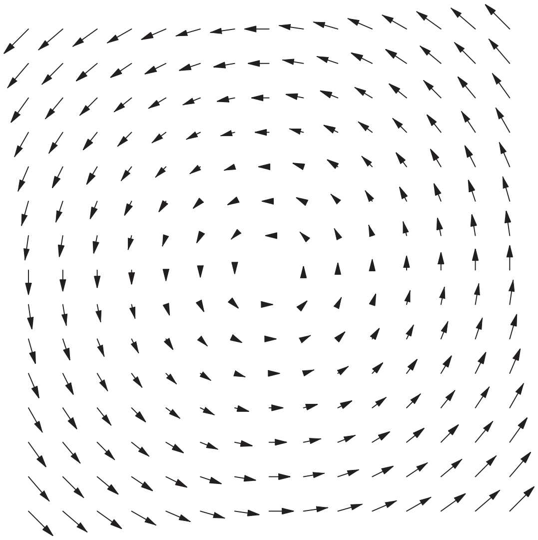
Problem 4.4.10. *Calculate the divergence of* $(-y, x)$.

Solution. By the exact same argument as the previous problem, the divergence of this guy is also 0 (see picture[6] below).



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[6] This is actually our standard "vector field with curl." It so happens that this field is divergenceless.

Problem 4.4.13. *Calculate the curl of* $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Solution. I have somewhat of an aversion to computing curls because it is a royal pain. So therefore I look at that vector field and using the un-gradienting technique of problem 30 (see below) and of problem 3 above, we see that it's the gradient of $\frac{1}{2}(x^2 + y^2 + z^2)$. Using the fact that $\nabla \times \nabla f = \mathbf{0}$, we find that the curl of this is $\mathbf{0}$. □

Problem 4.4.14. *Compute the curl of* $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

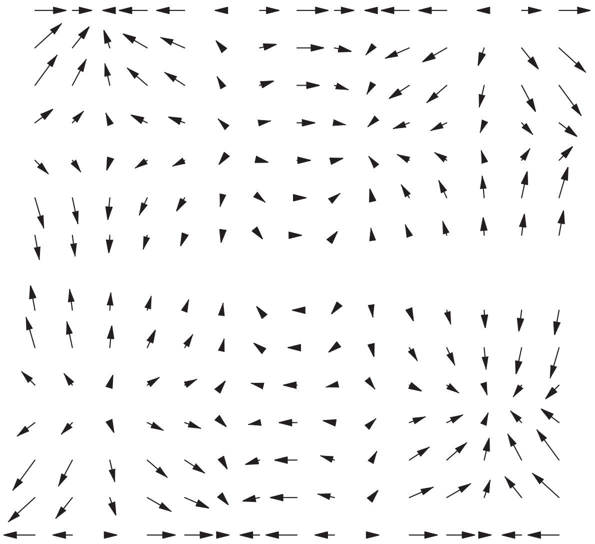
Solution. This is the same field as in the picture above for #2. Ask yourself, does that vector field look curly to you? Doesn't look like it to me. Let's have a calculation:

$$\left(\frac{\partial(xz)}{\partial y} - \frac{\partial(xy)}{\partial z}, \frac{\partial(xy)}{\partial z} - \frac{\partial(yz)}{\partial x}, \frac{\partial(yz)}{\partial x} - \frac{\partial(xz)}{\partial y}\right) = (0, 0, 0)$$

Alternatively you could see it by noticing that $\mathbf{F} = \nabla f$ where $f(x, y, z) = xyz$. To get you to practice the method of "ungradienting" we proceed as follows: suppose f is unknown such that $\nabla f = (yz, xz, xy)$. Then $\partial f/\partial x = yz$ so $f(x, y, z) = xyz + C(y, z)$ (partial integration with respect to x may well be a function of y and z); then $\partial f/\partial y = xz + C_y(y, z)$ which means, matching up with what we know ∇f really is, $C_y(y, z) = 0$. So C is possibly only a function of z. But then $\partial f/\partial z = xy + C_z(z)$ so that $C_z = 0$ and hence doesn't depend on z either. Therefore $f(x, y, z) = xyz + K$ where K is a genuine constant. So $\mathbf{F}$ is both a divergence-less and a curl-less field (but note that $(-y, x)$ is divergence-less but not curl-less so this is just a coincidental occurrence). □

Problem 4.4.25. *Show that* $\mathbf{F} = y\cos x\mathbf{i} + x\sin y\mathbf{j}$ *is not a gradient field*

Solution. First off let's look at a picture.



I can't tell heads or tails whether it has curl or not. It's quite strange. We'll have to resort to brute force curl computation, the kind that I don't like. Since it is 2D, we stick a zero in the 3rd slot and the curl will point in the $\mathbf{k}$ direction:

$$\nabla \times (y\cos x, x\sin y, 0) = \left(0, 0, \frac{\partial}{\partial x}(x\sin y) - \frac{\partial}{\partial y}(y\cos x)\right) = (0, 0, \sin y - \cos x) \neq \mathbf{0}.$$

Therefore it cannot be the curl of anything. For more information on curls of vector fields and its relation to the gradient, see the additional supplement I have on the web site. □

Problem 4.4.30. *Let* $\mathbf{r}(x, y, z) = (x, y, z)$ *and* $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$. *Show the following*

a. $\nabla(1/r) = -\mathbf{r}/r^3$ *for* $r \neq 0$, *and* $\nabla(r^n) = nr^{n-2}\mathbf{r}$ *and* $\nabla(\log r) = \mathbf{r}/r^2$.

b. $\nabla^2(1/r) = 0$[7] *(that's not zero to the 6th, that's a footnote); and in general* $\nabla^2 r^n = n(n + 1)r^{n-2}$

c. $\nabla \cdot (\mathbf{r}/r^3) = 0$ *and in general* $\nabla \cdot (r^n\mathbf{r}) = (n + 3)r^n$

d. $\nabla \times \mathbf{r} = \mathbf{0}$ *and in general* $\nabla \times (r^n\mathbf{r}) = 0$.

Solution. First off, this is a very long problem[8]. It turns out it is possible to kill many birds with one stone using the vector identities on page 307.

a. Again, you should have a sense of déjà vu (a glitch in The (Jacobian) Matrix) in that you've calculated $\nabla(1/r)$ twice before (possibly more in a physics class). Let's cut directly to the general case: notice first that $\nabla r^n = nr^{n-1}\nabla r$ by the Chain Rule. So it reduces to just calculating $\nabla r = \nabla\left(\sqrt{x^2 + y^2 + z^2}\right)$ which is easy to do because of the symmetry in x, y, and z:

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} 2x = \frac{x}{r}$$

from which it follows that

$$\nabla r = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{\mathbf{r}}{r}.$$

Combining it with the above, $\nabla(r^n) = nr^{n-1}\mathbf{r}/r = nr^{n-2}\mathbf{r}$, as promised. For the special case log (which is in a sense the case $n = 0$) we note $\log r = \frac{1}{2}\log(x^2 + y^2 + z^2)$ and hence $\partial/\partial x(\log r) = \frac{1}{2}1/(x^2 + y^2 + z^2)2x = x/r^2$. Therefore by symmetry we can calculate $\partial(\log r)/\partial y$ and $\partial(\log r)/\partial z$ by replacing x with y and z respectively, so that $\nabla(\log r) = (x, y, z)/r^2 = \mathbf{r}/r^2$.

b. $\nabla^2 r^n = \nabla \cdot (\nabla r^n) = \nabla \cdot (nr^{n-2}\mathbf{r})$. By using an identity[9] on page 307, we have $\nabla \cdot (nr^{n-2}\mathbf{r}) = \nabla(nr^{n-2}) \cdot \mathbf{r} + nr^{n-2}(\nabla \cdot \mathbf{r}) = n\nabla(r^{n-2}) \cdot \mathbf{r} + 3nr^{n-2}$ (note $\nabla \cdot \mathbf{r} = 3$ as it's just $\partial x/\partial x + \partial y/\partial y + \partial z/\partial z$). Look at the left term, however. Note that we already know how to calculate $\nabla(r^{n-2})$ by part *a*, namely it's $(n - 2)r^{n-4}\mathbf{r}$. So $\nabla \cdot (nr^{n-2}\mathbf{r}) = n(n - 2)r^{n-4}\mathbf{r} \cdot \mathbf{r} + 3nr^{n-2}$. But $\mathbf{r} \cdot \mathbf{r} = \|\mathbf{r}\|^2 = r^2$ by definition. So the r^2 gets absorbed into the r^{n-4} so we have, factoring out nr^{n-2}:

$$\nabla \cdot (nr^{n-2}\mathbf{r}) = n(n - 2)r^{n-2} + 3nr^{n-2} = nr^{n-2}(n - 2 + 3) = n(n + 1)r^{n-2}$$

as desired. Plugging in $n = -1$ we see the $n + 1$ gives 0 in the above so the answer is 0.

c. Note that $r^n\mathbf{r} = \frac{1}{n+2}(n + 2)r^n\mathbf{r} = \frac{1}{n+2}\nabla(r^{n+2})$ so that $\nabla \cdot (r^n\mathbf{r}) = \frac{1}{n+2}\nabla^2(r^{n+2}) = (n + 2)(n + 3)r^n/(n + 2) = (n + 3)r^n$ unless $n = -2$ where bad things may happen. But we recognize that as $\nabla^2(\log r)$, which is, after another routine calculation, in fact $-r^{-2}$.

Joke 4.4.1. An engineer sets a trash can on fire and takes a mathematician to to the burning scene. Quickly the mathematician gets a fire extinguisher, and puts the fire out. Next the engineer takes the mathematician to a trash can not on fire. The mathematician then goes and lights the trash can on fire, then puts it out. Puzzled, the engineer asks "Why in the world did you do that?!"

"Simple," says the mathematician. "I was reducing the problem to one that I have previously solved."

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[7] This says that $1/r$ is a *harmonic* function, one of the most important facts in physics and other applications; in particular this is the potential of the inverse square vector field—can you say "gravity" and "electrostatics"?

[8] "Humans have a tendency to state the very obvious." — Douglas Adams, *Hitchhiker's Guide to the Galaxy*

[9] $\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + (f)(\nabla \cdot \mathbf{F})$

d. It is useful to do the specific case first, because it is part of the calculation of the general case. $\nabla\times\mathbf{r} = \mathbf{0}$ because $\mathbf{r} = \nabla(\frac{1}{2}(x^2+y^2+z^2))$ (by a calculation entirely analogous to problem 3 in section 4.3). Using an analogous formula[10] from page 307 we have $\nabla\times r^n\mathbf{r} = \nabla(r^n)\times\mathbf{r}+r^n\nabla\times\mathbf{r} = \nabla(r^n)\times\mathbf{r}$ because the second term is $\mathbf{0}$ by the special case. But $\nabla(r^n)\times\mathbf{r} = nr^{n-2}\mathbf{r}\times\mathbf{r}$ by part a). But $\mathbf{r}\times\mathbf{r} = \mathbf{0}$ as any vector crossed with itself is $\mathbf{0}$. Therefore the whole sum is $\mathbf{0}$.

□

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[10] $\nabla\times(f\mathbf{F}) = \nabla f\times\mathbf{F} + (f)(\nabla\times\mathbf{F})$