Homework 5 Solution

Math 20E
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5 Chapter 5

5.1 Problems 5.1: 1a, 2a, 4

Problem 5.1 (1a and 2a). *Evaluate* $\int_{-1}^{1}\int_{0}^{1}(x^4y+y^2)\,dydx$ *and* $\int_{0}^{1}\int_{-1}^{1}(x^4y+y^2)\,dx\,dy$.

Solution. A basic problem involving interchanging the order of integration. For the first one, we have

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{1}(x^4y+y^2)\,dydx &= \int_{-1}^{1}\left[x^4\frac{y^2}{2}+\frac{y^3}{3}\right]_{y=0}^{1}\,dx\\ &= \int_{-1}^{1}\left(\frac{x^4}{2}+\frac{1}{3}\right)dx\\ &= \left[\frac{x^5}{10}+\frac{x}{3}\right]_{x=-1}^{1}\\ &= 2\left(\frac{1}{10}+\frac{1}{3}\right)\\ &= \frac{13}{15}\end{aligned}$$

where we used symmetry and the oddness of the functions x^5 and x to get that second-to-last inequality (it is my firm belief that the less subtractions you have to do, the better). The second way:

$$\begin{aligned}\int_{0}^{1}\int_{-1}^{1}(x^4y+y^2)\,dxdy &= \int_{0}^{1}\left[y\frac{x^5}{5}+xy^2\right]_{x=-1}^{1}\,dy\\ &= \int_{0}^{1}2\left(\frac{y}{5}+y^2\right)dy\\ &= 2\left[\frac{y^2}{10}+\frac{y^3}{3}\right]_{y=0}^{1}\\ &= 2\left(\frac{1}{10}+\frac{1}{3}\right)\\ &= \frac{13}{15}\end{aligned}$$

where symmetry was used again but this time in the second equality. The answers, as we expect, are the same. □

Problem 5.4. *Use Cavalieri's Principle to compute the volume of a wiggly square tube thing depicted in 5.1.11.*

Solution. The point of Cavalieri's Principle is to be able to calculate volumes using only facts about the cross-sectional area and not how the thing wiggles in space. The only thing that matters is what total (straight-line) vertical distance you have. In this case the cross-sectional area is always a 3×5 rectangle so the area is the constant function 15. We integrate that for 7 vertical units and that gives us $15\cdot 7=105$. □

5.2 Problems 5.2: 2d, 4

Problem 5.2. *Evaluate the following double integral for $R = [0,1] \times [0,1]$:*

$$\iint_R (x^2 + 2xy + y\sqrt{x}) dA$$

Solution. Note that R is just a rectangle so this is not really different from the problems before.

$$\begin{aligned}
\iint_R (x^2 + 2xy + y\sqrt{x})\, dA &= \int_0^1 \int_0^1 (x^2 + 2xy + y\sqrt{x})\, dx\, dy \\
&= \int_0^1 \left[\frac{x^3}{3} + x^2 y + \frac{2}{3} y x^{3/2}\right]_{x=0}^1 dy \\
&= \int_0^1 \left(\frac{1}{3} + y + \frac{2}{3} y\right) dy \\
&= \left[\frac{y}{3} + \frac{5y^2}{3 \cdot 2}\right]_{y=0}^1 \\
&= \frac{1}{3} + \frac{5}{6} \\
&= \frac{7}{6}.
\end{aligned}$$

It is also instructive to do it the other way.

$$\begin{aligned}
\iint_R (x^2 + 2xy + y\sqrt{x})\, dA &= \int_0^1 \int_0^1 (x^2 + 2xy + y\sqrt{x})\, dy\, dx \\
&= \int_0^1 \left[x^2 y + y^2 x + \frac{1}{2} y^2 x^{1/2}\right]_{y=0}^1 dx \\
&= \int_0^1 \left(x^2 + x + \frac{1}{2} x^{1/2}\right) dx \\
&= \left[\frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{3} x^{3/2}\right]_{x=0}^1 \\
&= \frac{1}{3} + \frac{1}{2} + \frac{1}{3} \\
&= \frac{7}{6}.
\end{aligned}$$

□

Problem 5.3. *Calculate the volume of the solid bounded by the xz-plane, yz-plane, xy-plane, $x = 1$, $y = 1$, and the surface $z = x^2 + y^4$.*

Solution. Note that the equations of the xz-plane, yz-plane, xy-plane are $y = 0$, $x = 0$, $z = 0$ respectively. Along with $x = 1$ and $y = 1$ this says the domain in the plane we're talking about is $0 \le x \le 1$ and $0 \le y \le 1$, that is $[0,1] \times [0,1]$. The other two surfaces $z = 0$ and $z = x^2 + y^4$ suggest that we are to compute the volume under the graph over the rectangle. This is exactly analogous to computing the area under the curve in 1-variable calculus—we integrate the top surface:

$$\begin{aligned} V &= \iint_{[0,1]\times[0,1]} (x^2 + y^4)\, dA \\ &= \int_0^1 \int_0^1 (x^2 + y^4)\, dx\, dy \\ &= \int_0^1 \left[\frac{x^3}{3} + xy^4\right]_{x=0}^1 dy \\ &= \int_0^1 \left(\frac{1}{3} + y^4\right) dy \\ &= \left[\frac{y}{3} + \frac{y^5}{5}\right]_{y=0}^1 \\ &= \left(\frac{1}{3} + \frac{1}{5}\right) \\ &= \frac{8}{15} \end{aligned}$$

I'll leave the computation in the other direction as a dreaded exercise for the reader. □

5.3 Problems 5.3: 2b, 6

The general concept for setting up integration bounds for funny regions is that, whatever you are integrating with respect to, you are integrating parallel to that variable's axis—integrating with respect to dx means parallel to the x-axis, namely along horizontal segments. Then it's all a matter of writing out in formulas what the lower extent and the upper extent of that variable (in terms of the other variables, which are being held fixed—just like partial derivatives, when integrating, we regard the other guys as constant). Enough talk, the problems will be very illustrative.

Problem 5.2. *Evaluate the integral and sketch the corresponding region of integration for the following:*

$$\int_{-1}^{1} \int_{-2|x|}^{|x|} e^{x+y}\, dy\, dx$$

Note that $|x|$ looks like 2 diagonal lines meeting at the origin (a letter V) and so therefore $-2|x|$ looks like it flipped over and narrower (like the greek letter Λ (lambda) in skinny font). More straightforward is the guy with the x-coordinate, the vertical lines $x = -1$ and $x = 1$. Therefore the region looks like a bow-tie which is kind of stretched out at the bottom (see Figure 1).

The thing is about this integral is that it is really done using a piecewise function. Therefore to

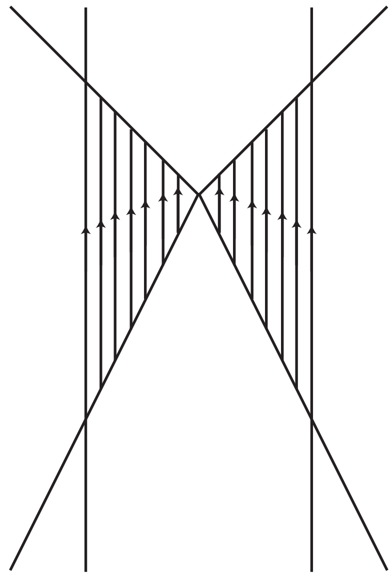

Figure 1: Region of integration for Problem 2b

evaluate it, we add the cases −1 to 0 and 0 to 1 separately:

$$\begin{aligned}
\int_{-1}^{1}\int_{-2|x|}^{|x|} e^{x+y}\,dy\,dx &= \int_{-1}^{0}\int_{2x}^{-x} e^{x+y}\,dy\,dx + \int_{0}^{1}\int_{-2x}^{x} e^{x+y}\,dy\,dx \\
&= \int_{-1}^{0} [e^{x+y}]_{y=2x}^{-x}\,dx + \int_{0}^{1} [e^{x+y}]_{y=-2x}^{x}\,dx \\
&= \int_{-1}^{0} \left(e^{0} - e^{3x}\right)\,dx + \int_{0}^{1} \left(e^{2x} - e^{-x}\right)\,dx \\
&= \left[x - \frac{1}{3}e^{3x}\right]_{-1}^{0} + \left[\frac{1}{2}e^{2x} + e^{-x}\right]_{0}^{1} \\
&= 0 - \frac{1}{3} - (-1) + \frac{e^{-3}}{3} + \frac{e^{2}}{2} + e^{-1} - \frac{1}{2} - 1 \\
&= \frac{e^{2}}{2} - \frac{5}{6} + e^{-1} + \frac{e^{-3}}{3}.
\end{aligned}$$

Problem 5.6. *Let D be the region bounded by the positive x- and y-axes and the line $3x + 4y = 10$. Compute*

$$\iint_D (x^2 + y^2)\,dA$$

Solution. The region is depicted in Figure 2—$3x + 4y = 10$ describes a straight line of slope $-3/4$ with y-intercept $\frac{5}{2}$ and x-intercept $\frac{10}{3}$. The region of integration is then the triangle bounded by $x = 0$, $y = 0$, and this line. If we integrate with respect to y first, then we are integrating along the vertical segments (left figure), and so y ranges from 0 to up to the line. We must express this line in terms of x to continue, taking note of the fact that the limit of integration must explicitly define y in terms of the other variables (more on this in the next section). “Explicitly” just means that y has to be completely isolated on one side of the equation. So writing the equation of the line, we have $3x + 4y = 10$ means

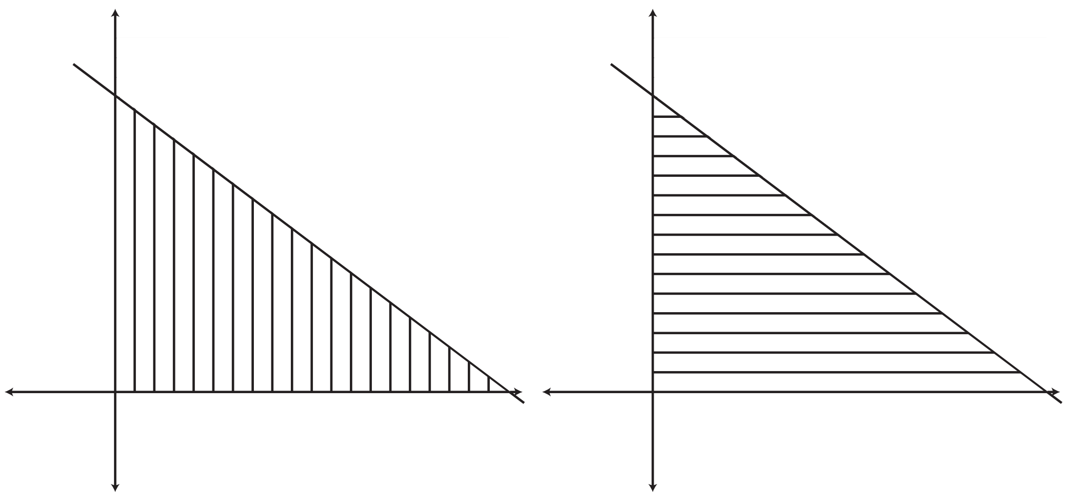

Figure 2: Region of integration for Problem 6, left: with respect to y first; right: with respect to x first.

$4y = 10 - 3x$ or $y = -\frac{3}{4}x + \frac{5}{2}$. After getting rid of the y-variable we can project down to the line, to get x ranging from 0 to whenever the line meets the x-axis, namely when $y = 0$ or $x = 10/3$. So therefore we have

$$\begin{aligned}\iint_D (x^2 + y^2)\, dA &= \int_0^{10/3} \int_0^{-\frac{3}{4}x+\frac{5}{2}} (x^2 + y^2)\, dy\, dx \\ &= \int_0^{10/3} \left[x^2 y + \frac{y^3}{3}\right]_{y=0}^{y=-\frac{3}{4}x+\frac{5}{2}} dx \\ &= \int_0^{10/3} \left(-\frac{3}{4}x^3 + \frac{5}{2}x^2 + \frac{1}{3}\left(-\frac{3}{4}x + \frac{5}{2}\right)^3\right) dx\end{aligned}$$

This should be sufficient—the point of the problem is the setup, not the evaluation and the subsequent ugly computation (this is not 20b). Basically I'd split it up and take care of the last one with a u-substitution. It is now instructive to do the integral in the other order: we now refer to the right figure as we integrate with resepect to x first. x goes from $x = 0$ to the line, which we now express explicitly in terms of y: $3x + 4y = 10$ means $3x = 10 - 4y$ or $x = -\frac{4}{3}y + \frac{10}{3}$.

$$\begin{aligned}\iint_D (x^2 + y^2)\, dA &= \int_0^{5/2} \int_0^{-\frac{4}{3}y+\frac{10}{3}} (x^2 + y^2)\, dx\, dy \\ &= \int_0^{5/2} \left[\frac{x^3}{3} + xy^2\right]_{x=0}^{x=-\frac{4}{3}y+\frac{10}{3}} dy \\ &= \int_0^{5/2} \left(\frac{1}{3}\left(-\frac{4}{3}y + \frac{10}{3}\right)^3 - \frac{4}{3}y^3 + \frac{10}{3}y^2\right) dy\end{aligned}$$

which I will once again leave as a dreaded exercise for the reader (bribe a 20b student to do the rest if you like. Better yet, bribe 2 of them, have one do the previous one and another to do this, and see if they get the same answer). □

5.4 Problems 5.4: 2c, 8

Ok this is where integration gets very messy. When you have to integrate regions with variables in the upper limits and then switch them around. The main problem is, when interchanging the order of integration, you can't just switch them blindly as you can in the case of integrating over a rectangle. You have to change the formulas for the bounds, because when you integrate with respect to a variable, you are making it disappear completely. Consequently you have a few rules of thumb: as you move progressively outward, the bounds contain fewer variables; namely they cannot contain variables which have already been integrated (i.e. appeared with a d in an inner integral). A consequence of this is that the outermost integral must have constants as bounds—no function of any variable at all.

5.5 Problems 5.5: 4, 11