Math 20E Midterm 2 Study Notes

CHRIS TIEE

1 Multiple Integration: Overview of Concepts

The story so far: In the Beginning the Integral was created. This made (and still makes) a lot of people very angry and was (and still is) widely regarded as a bad move (curse you, Newton! Curse you!!!). Ok, not that story. The point of the material covered since the last midterm (Homeworks 5, 6, 7, 8) has been to extend the idea of integration over higher-dimensional domains. We'd like to integrate over domains that aren't always nice and straight, but rather, regions that have curvy boundaries and even regions with curvy interior. With the mass of different notation, techniques, etc. it is easy to get confused. You shouldn't feel too bad if you are a bit lost. Some burning questions you might have are "What the heck is a surface integral? Vector surface integral? What's a Jacobian and why do we need it, and why does this $\|\mathbf{T}_u \times \mathbf{T}_v\|$ make a sudden, random appearance?" I'll try to give a summary of all these things, as it is always helpful to try to get the big picture.

1.1 Chapter 5

The material of chapter 5 is integration over flat 2 and 3-dimensional domains, but even though they are flat on the inside, they can have funny boundaries. This issue never happens in 1 dimension because the boundary of an interval is always going to be 2 points![1] But 2-dimensional domains can have 1-dimensional boundaries, and these can wiggle around. This wiggle-around business corresponds to making the bounds of integration have complicated bounds that are *functions* of the variables that haven't already been integrated.

The regions of integration are usually described by some kind of inequalities: x lies in between such and such, then y also lies between such and such, which may also depend on what x is, etc. What the book describes as a "simple region" is that we can choose one variable independently, say x to lie in between two *constants*, and then the other variable can lie in between two numbers depending on the first variable only. Often we can actually convert from the description in one variable to the other. Like a disk in the plane (area bounded by a circle), which is usually written $\{(x, y) : x^2 + y^2 \leq 1\}$ can also be represented as the region $\{(x, y) : -1 \leq x \leq 1 \text{ and } -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}\}$, or $\{(x, y) : -1 \leq y \leq 1 \text{ and } -\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2}\}$. The heart of the problem of interchanging the order of integration is basically converting from one description of the domain to the other.

1.2 Chapter 6

Chapter 6 introduces a way to warp some kinds of domains with boundaries that are hard to work with, into domains that are easier to handle. For example, integration over rectangles is always nice because the bounds of integration are constants. Also, it may significantly simplify the integrand (function to be integrated) itself. This leads to the idea of "transformations"—functions taking a domain and changing it into something else. The confusing issue here is which way we're going—since transformations can go forward and back. In other words, what the heck are D and D^*. So the idea is, we assume that as given, our problem is described in the (x, y) or (x, y, z) domain and looks complicated.

---

[1] Actually, in the study of Lebesgue integration—covered in the class I'm taking right now—we can integrate over crazy weird domains even in one dimension; instead of intervals, for example, we can integrate over a scattered dust of points. Thank goodness we usually don't have to actually *calculate* these things!

The idea is if we can express $(x, y) = T(u, v)$ for some transformation T, and the domain where it comes from can be expressed more simply, then we rewrite the domain D of the the given variables (x, y) in terms of the domain D^* of the new variables (u, v). So really, computing the domain is running the T backwards from (x, y) to get (u, v). More relevant to computation: if we have $D = \{(x, y) : a \leq x \leq b \text{ and } f(x) \leq y \leq g(x)\}$ (a domain that is simple with respect to x), and we have $(x, y) = T(u, v) = (T_1(u, v), T_2(u, v))$, then, to compute D^* we substitute the x and y directly with $T_1(u, v)$ and $T_2(u, v)$ directly in the expression. So $D^* = \{(u, v) : a \leq T_1(u, v) \leq b \text{ and } f(T_1(u, v)) \leq T_2(u, v) \leq g(T_1(u, v))\}$. Then this satisfies $T(D^*) = D$ or equivalently $D^* = T^{-1}(D)$ the inverse function. The integrand will also be a function in terms of (x, y) so we also change it to be in terms of (u, v) as well. Finally, the Jacobian is necessary as a "correction factor"—integration is, at its heart, a system of measurement. In particular, it's all about area and volume. So if our transformation warps space in a funny way, it will change area. The Jacobian corrects this and is analogous to the derivative of a composition of functions $(f \circ g)'(t)$ needing the extra $g'(t)$ in is chain rule expansion $f'(g(t))g'(t)$. In other words, to finally get to the point, this method of integration is the multivariable "u-substitution"—except now it's really (u, v)- (or even (u, v, w)-) substitution. It's summed up on the next formula:

$$\iint_D F(x, y)\, dx\, dy = \iint_{D^*=T^{-1}(D)} F(T(u, v)) \det[\mathbf{D}T(u, v)]\, du\, dv$$

A "real" example will follow in a moment.

The most well-known, and indeed most useful, are the coordinate transformations introduced the 2nd or 3rd week—polar coordinates (for 2-D problems) and cylindrical and spherical (for 3-D problems), which are $P(r, \theta) = (r\cos\theta, r\sin\theta)$, $C(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$, and $S(\rho, \varphi, \theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$. An illustrative example, then, is $\iint_{x^2+y^2\leq 1} e^{x^2+y^2}\, dx\, dy$. Recognizing $x^2 + y^2$, because we know this is r^2, it swells in our hearts to convert to polar coordinates. So how do we re-express the domain $D = \{(x, y) : x^2 + y^2 \leq 1\}$ in terms of r and θ? We plug in what x is in terms of it, wherever it occurs, and similarly for y: $D^* = \{(r, \theta) : r^2\cos^2\theta + r^2\sin^2\theta \leq 1\}$. Recognizing $\sin^2\theta + \cos^2\theta = 1$ we have $D^* = \{(r, \theta) : r^2 \leq 1\}$ or since r is always positive, $D^* = \{(r, \theta) : 0 \leq r \leq 1\}$. But what about θ? If no restriction is made on θ, its "natural range" is $[0, 2\pi]$. The natural range of r which you will usually not have occasion to use unless you do improper integrals, is $0 \leq r < \infty$. In most cases you will end up explicitly computing a range for r and leave θ alone. Finally, the Jacobian:

$$\mathbf{D}T(r, \theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -rsin\theta & r\cos\theta \end{bmatrix}$$

which has a determinant of $r\cos^2\theta + r\sin^2\theta = r$. So the integral is now

$$\iint_{r\leq 1} e^{r^2} r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 e^{r^2} r\, dr\, d\theta$$

Now that this beast has been reduced to two 1-dimensional integrals, you can bribe your friends in 20B to finish it off for you (except, of course, on the exam, but you won't encounter nasty integrals there, anyway). For the interested, it's a 1-variable u-substitution. For your reference the Jacobians for Cylindrical and Spherical are, respectively r and $\rho^2\sin\varphi$. These are worth memorizing.

1.3 Chapter 7

The subject of interest here is integration over domains where, not only the boundaries can be curvy, but the interiors can curve around, too. And here, things get interesting even in 1 dimension, since the interior of a 1-dimensional thing can curve around (as anyone who's ever tied shoelaces knows). One might think that totally different methods are needed here. Different, yes. Totally, no. We need a way to speak of the domain of integration. This is usually given as a parameterization (a word I only since recently have been able to pronounce without stumbling over something). A parameterization is actually just a transformation (like in Chapter 6) except the range (where you end up) of the transformation can have more dimensions than the domain (where you start from)—this expresses the fact that the flat thing is being put into a dimension higher up, which allows it to wiggle around. I will talk about actually finding parameterizations—half the battle in doing line and surface integrals—in an upcoming section, with a couple of good examples.

In integration over these surfaces, therefore, we find a transformation from the line or plane to this object that wiggles in a higher dimension. It turns out the only way you can actually compute the integral is to use the transformation (parameterizaton) to *convert the integral into one that can be done in flat space*. Then we've reduced it to Chapters 5 and 6. This is the essential relation between Chapter 7 and Chapters 5 and 6. Parameterizations really being transformations in disguise, then, raises the question: what is the Jacobian? It is the $\|\mathbf{T}_u \times \mathbf{T}_v\|$ (or in the case for curves, $\|\mathbf{c}'(t)\|$). These are the only kinds of parameterized domains we consider in this class.[2]

Also in this chapter we are introduced to two different types of integrals over curvy domains: path vs. line integrals, and scalar vs. vector surface integrals. Actually it is the path/scalar integral that is the basic thing. Integrals of vector fields over curvy domains are just one very useful application of this concept—we integrate a particular scalar function that comes from the dot product of the vector function with some other vector field. However, in the process, a computational shortcut comes up that makes life easier, but at the same time can be confusing because you end up thinking that it is yet another different thing to do. Paradoxically integrals of vector fields (which sound harder) end up being easier to compute because of this shortcut—it eliminates that square root term (and as we all know, square roots cause big trouble when it comes to integration). What happens is this: we usually integrate a vector field dot-producted with the unit tangent vector (for curves), or the unit normal vector (for surfaces), and these correspond to $\mathbf{t} = \mathbf{c}'(t)/\|\mathbf{c}'(t)\|$ and $\mathbf{n} = (\mathbf{T}_u \times \mathbf{T}_v)/\|\mathbf{T}_u \times \mathbf{T}_v\|$. Then when doing the normal surface integral, the denominator cancels the "Jacobian" term. To sum up in formulas, if C is a curve parameterized by $\mathbf{c}(t)$, S is a surface parameterized by $\mathbf{T}(u, v)$, and $\mathbf{F}$ a vector field,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \mathbf{t}\, ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|} \|\mathbf{c}'(t)\|\, dt = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

where $[a, b]$ is the domain of $\mathbf{c}$, and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F}(\mathbf{T}(u, v)) \cdot \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv = \iint_D \mathbf{F}(\mathbf{T}(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$$

where D is the domain of $\mathbf{T}$. To summarize in less scary-looking formulas, the heart of line and

---

[2] If anyone ever goes and parameterizes 3-space object (u, v, w) to wiggle in 4-space, it would be

$$\left\| \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \boldsymbol{\ell} \\ \partial T_1/\partial u & \partial T_2/\partial u & \partial T_3/\partial u & \partial T_4/\partial u \\ \partial T_1/\partial v & \partial T_2/\partial v & \partial T_3/\partial v & \partial T_4/\partial v \\ \partial T_1/\partial w & \partial T_2/\partial w & \partial T_3/\partial w & \partial T_4/\partial w \end{bmatrix} \right\|$$

It's even more fun parameterizing 2D surfaces in 4-space (this requires 6 (gasp!) components!).

surface integrals lies in the ds's and dS's, the central four formulas are:

$$ds = \|\mathbf{c}'(t)\|\, dt \tag{1.1}$$

$$d\mathbf{s} = \mathbf{t}\, ds = \mathbf{c}'(t)\, dt \tag{1.2}$$

$$dS = \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv \tag{1.3}$$

$$d\mathbf{S} = \mathbf{n}\, dS = (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv \tag{1.4}$$

1.4 Other random conceptual notes

You will probably encounter problems asking about area of a region in the plane or area of a surface in space (and similarly, volume of a region in space, and arc length for curves). Integrals, as mentioned before, are about measurement—lengths, areas, volumes (the class in analysis I am taking is concerned with a lot of *measure theory* which is basically an abstract, super-fortified theory of integration). The most basic example of something you could compute with integrals, then, are areas. And indeed, in the beginning way back when, you learned about the integral as "the area under the curve." Going one dimension up, we can integrate the volume under a surface. But this is only a special case of the integral as measurement in one dimension up. If you integrate the constant function 1 (i.e. no integrand at all), then it will give you the measure of the domain of integration—whether it is flat, has curvy boundaries, or has curvy interior. Indeed the "area under the curve" interpretation of a one-variable integral $\int_a^b f(x)\, dx$ is really the area of the region in the plane defined by $a \le x \le b$, and $0 \le y \le f(x)$. so that we can simply recast the integral

$$\int_a^b f(x)\, dx = \int_a^b \int_0^{f(x)} 1\, dy\, dx$$

(to make a simple thing more complicated =). Also $\int_a^b 1\, dx = b - a$ as I hope you can compute, which is the length of $[a, b]$. Similarly, area of a region is the double integral of 1 over that region, surface area of a parameterized surface is the surface integral of 1, length of a curve is the path integral of 1, and volume of a region in space is the triple integral of 1. Now when we have an integrand which is some function G, this is really assigning a kind of "weighting function" which somehow makes the area in one part of the domain "count more" than the other. And if G becomes negative some parts count as negative and actually subtract from the total contribution. This is why, for example, we can calculate things like masses from "density functions"—because this says, really, parts of your domain are denser than others and can contribute more to the total mass of the region you're calculating over. This is the big, conceptual thing about integration, and it is the reason why the Integral was invented (despite the fact that it still makes a lot of people angry).

2 Examples and Applications

This section forthcoming. . .