A Quick Note on Visualizing Ricci Flow

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1 Introduction

> Although it is usually simpler to prove a general fact than to prove numerous special cases of it, for a student the content of a mathematical theory is never larger than the set of examples that are thoroughly understood. -- Vladimir I. Arnold

Despite the usually high-dimensional and abstract nature of Ricci Flow (remember, the curvature tensor only really becomes distinguished from its simpler, averaged-out quantities, the Ricci and scalar curvatures, in dimensions greater than 3), it nevertheless should be instructive to visualize low-dimensional examples to help get a feel for things. In most lectures on Ricci flow, one draws informal diagrams to show how curvature evolves. But at some point one wonders, how "accurate" are these pictures? What is *really* happening? One might think this is just some idle speculation, but being able to accurately see what, for example, "pinching" might entail, or when a solution might "extend" to include additional points, yields very important information. Unfortunately, explicitly computable examples are difficult to find; but the knowledge gained from those few visualizable ones is still valuable, because they frequently arise as limit solutions.

2 "Physical Realization" via Embedding

The first problem we have to recognize is that because of the intrinsic nature of Ricci Flow, we regard everything taking placed in a fixed manifold, evolving a metric, which is an additional structure on this manifold. Essentially what is happening is we are specifying how our rulers and protractors are getting deformed. We can of course visualize this by showing how a coordinate grid gets deformed.

But of course, this is not nearly as dramatic as the so-called "active" viewpoint. What if we regard the rulers and protractors as fixed? That is essentially what we do when we draw heuristic diagrams of what is going on in a pinching, or pictures of cigars. But we can get precise about this. What does it mean to fix the rulers and protractors? It means that we are considering *isometric embeddings* of the manifold into some ambient space with a fixed metric. Let's make this very explicit.

2.1 Definition. Let $g(s)$ be a (smooth) 1-parameter family metrics on a manifold M^m, and (N^n, h) be another Riemannian manifold with a fixed metric h. A *realization* of $(M, g(s))$ in (N, h) is a 1-parameter family of isometric embeddings $\iota(s) = \iota_s : M \hookrightarrow N$, i.e. a family of embeddings such that

$$\iota(s)^* h = g(s).$$

Of course, we will be primarily interested in the case that $N = \mathbb{R}^3$ and h is the Euclidean metric, but to give this note some kind of air of relevance with fancy-schmancy modern terminology, we've given this (somewhat intuitive, actually) precise, general definition. We'll call a realization with $N = \mathbb{R}^n$, h the Euclidean metric, a *physical realization*.

Recall that, despite the name, an isometric embedding might not actually preserve distances, since geodesics in the ambient space may very well be different from those in the domain (just think of distance on a sphere vs. in 3-space i.e. great-circle vs. Euclidean distances; yet nobody would doubt that the standard embedding with the standard metric is an isometric one). So, what exactly can we count on to be preserved, besides the literal inner product? Here's one thing that should firmly ground the intuition: *If γ is a (smooth) curve* in M *(and thus $\iota_t \circ \gamma$ is the image curve under the embedding), then its length as computed via a line integral in M, and the length of $\iota_t \circ \gamma$ calculated as line integral in N, are the same.* The proof is just trivial calculation:

$$\begin{aligned}(2.1)\quad \mathscr{L}_{g(s)}[\gamma] = \int_a^b g(s)(\gamma'(t), \gamma'(t))\, dt &= \int_a^b (\iota_s^* h)(\gamma'(t), \gamma'(t))\, dt \\ = \int_a^b h(\iota_{s*}\gamma'(t), \iota_{s*}\gamma'(t))\, dt &= \int_a^b h((\iota_s \circ \gamma)'(t), (\iota_s \circ \gamma)'(t))\, dt = \mathscr{L}_h[\iota_s \circ \gamma].\end{aligned}$$

So the distance between two points *along a given path* is preserved in an isometric immersion, but not the Riemannian distance which is given as an *infimum* over all paths connecting the points, rather than specific fixed paths that correspond under the embedding.

When the metric on M is evolving, therefore, the length of γ in M is going to be a function of time, whereas the length of the embedded "physically realized" curve $\iota_t \circ \gamma$ is going to be a curve varying in time, whose length is computed via the fixed ambient metric h. We have realized a fixed curve in M, with measurement method changing, as a changing curve in N, with measuement method fixed. *It is this observation that makes precise the notion of "changing the rulers and protractors".*

3 The Shrinking Sphere

Enough talk; it's time for some action. Perhaps the simplest example which anyone who ever dipped their feet into Ricci flow has ever encountered, is the famous Shrinking Sphere. Ok, I don't know whether it really is that famous, but its apparent simplicity leaves little room for doubt that it is. Consider the 2-sphere of radius 1, S^2 with the usual geographic metric $g_c = d\varphi^2 + \sin^2(\varphi)d\theta^2$. Now let $g(t) = (1-2t)g_c$. Given that we know that the metric induced from $\mathbb{R}^3$ for a sphere of radius r is $r^2 g_c$, we might guess that our embedding should be given by $i_t(p) = p\sqrt{1-2t}$, or in terms of parametrizations, $j_t(\varphi,\theta) = (\sqrt{1-2t}\sin(\varphi)\cos(\theta), \sqrt{1-2t}\sin(\varphi)\sin(\theta), \sqrt{1-2t}\cos(\varphi))$. Note that considering a parametrization is simply embedding a part of the plane with a funky metric, i.e. the coordinate representation of (almost all) of S^2, into 3-space, so we're not really introducing anything really new here; in particular, $j_0^*(dx^2 + dy^2 + dz^2)$ is exactly g_c regarded as a metric on $(0,\pi)\times(0,2\pi)$ with the same coordinates. This is a solution to Ricci flow; we simply note that $\mathrm{Rc} = Kg$ where K is the Gaussian curvature, equal to $(1-2t)^{-1}$. Therefore $-2\,\mathrm{Rc}(g(t)) = -2g(t) = -2(1-2t)^{-1}(1-2t)g_c = -2g_c = \frac{\partial g}{\partial t}$.

To verify that it actually works, we just compute $j_t^*(dx^2 + dy^2 + dz^2)$, which is a very routine verification: $dx = d(\sqrt{1-2t}\sin\varphi\cos\theta) = \sqrt{1-2t}\cos\varphi\cos\theta d\varphi - \sqrt{1-2t}\sin\varphi\sin\theta d\theta$, so $dx^2 = (1-2t)\cos^2(\varphi)\cos^2(\theta)d\varphi^2 + (1-2t)\sin^2(\varphi)\sin^2(\theta)d\theta^2 - \cos\varphi\cos\theta\sin\varphi\sin\theta d\varphi d\theta$, and similarly $dy^2 = (1-2t)\cos^2(\varphi)\sin^2(\theta)d\varphi^2 + (1-2t)\sin^2(\varphi)\cos^2(\theta)d\theta^2 + \cos\varphi\cos\theta\sin\varphi\sin\theta d\varphi d\theta$ and $dz^2 =$

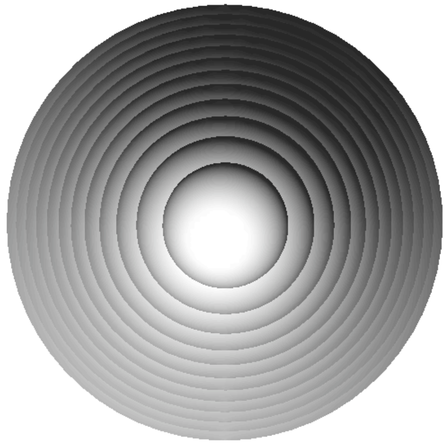

Figure 1: The Shrinking Sphere, cutaway view.

$(1-2t)\sin^2(\varphi)d\varphi^2$. Adding everything up, the cross terms cancel and we have

$$j_t^*(dx^2+dy^2+dz^2) = (1-2t)(\cos^2(\varphi)\cos^2(\theta)+\cos^2(\varphi)\sin^2(\theta)+\sin^2(\varphi))d\varphi^2+(1-2t)\sin^2(\varphi)(\sin^2(\theta)+\cos^2(\theta))d\theta^2 = (1-2t)d\varphi^2+(1-2t)\sin^2\varphi d\theta^2 = (1-2t)g_c, \tag{3.1}$$

as promised. What does this mean for visualization? If we imagine a sphere shrinking in time, the radius getting smaller proportional to the square root of the remaining time to the singularity ($t = 1/2$), we see that, since $\sqrt{x}$ has a steeper and steeper slope as $x \to 0$, the shrinking gets faster and faster as it gets smaller. Figure 1 shows a nested cutaway view of the spheres for several evenly-spaced times up to the singularity. Notice how the shrinking accelerates as $t \to 1/2$.

4 Rotationally Symmetric Solutions and Surfaces of Revolution

Now of course our example of the shrinking sphere yielded an easily guessed embedding, but perhaps we were lucky. In fact, we were; these embeddings are hard to find. Nevertheless, we can at least provide a somewhat explicit method of construction for 2-dimensional metrics that are rotationally symmetric, namely those of the form $\eta(s)^2ds^2+\psi(s)^2d\theta^2$ where s and θ are some coordinates and the metric coefficients η, ψ only depending on s. θ usually denotes some angular coordinate, as the name *rotationally symmetric* might suggest. It also suggests a method of search for parametrizations, namely *surfaces of revolution* in $\mathbb{R}^3$. The (restriction of the) Euclidean metric $\mathbb{R}^3$, to such a surface always yields rotationally symmetric metrics (unfortunately, however, to note that not all rotationally symmetric metrics actually arise from surfaces of revolution; not to worry, though, all the important examples in 2D we'll look at arise this way).

4.1 Surfaces of Revolution

We recall the formula for a surface of revolution (see [Pet] for lots of interesting calculations involving these): given a curve in the plane parametrized by $\gamma(t) = (\rho(t), \zeta(t))$, the surface formed by revolving this curve about the z-axis is given by $\Phi(t, \theta) = (\rho(t)\cos\theta, \rho(t)\sin\theta, \zeta(t))$. The inherited Euclidean metric is

$$
\begin{aligned}
(4.1)\quad \Phi^*(dx^2 + dy^2 + dz^2) &= d(\rho(t)\cos(\theta))^2 + d(\rho(t)\sin(\theta))^2 + d(\zeta(t))^2 \\
&= (\dot\rho(t)^2\cos^2\theta + \dot\rho(t)^2\sin^2\theta + \dot\zeta(t)^2)dt^2 + (\rho(t)^2\cos^2\theta + \rho(t)^2\sin^2\theta)d\theta^2 \\
&= (\dot\rho(t)^2 + \dot\zeta(t)^2)dt^2 + \rho^2(t)d\theta^2.
\end{aligned}
$$

So given $\eta(s)^2 ds^2 + \psi(s)^2 d\theta^2$ we set $\rho(s) = \psi(s)$, then calculate $\dot\rho(s)^2$. We now have to find $\zeta(s)$ such that $\dot\zeta(s)^2 + \dot\rho(s)^2 = \dot\zeta(s)^2 + \dot\psi(s)^2 = \eta(s)^2$, or $\dot\zeta(s) = \sqrt{\eta(s)^2 - \dot\psi(s)^2}$. Integrating from some convenient place we'll just call a, we set

$$\zeta(s) = \int_a^s \sqrt{\eta(u)^2 - \dot\psi(u)^2}\, du.$$

Of course, it might not be possible to carry this integral out, if for example, if $\dot\psi(u)^2 > \eta(u)^2$ at some u, which would make the integrand complex. This is is one way that some rotationally symmetric metric might fail to arise from a surface of revolution. Even then, it usually won't be possible to give the integral in elementary terms (the presence of a square root over some nasty functions is never very encouraging), and in fact this will be the case even for the examples we'll study. However there we'll show that it actually exists, and the numerical integration can be done reasonably well on a computer (*Mathematica* and Apple's Grapher had no trouble with the examples).

Since the metric $g(t) = (1 - 2t)g_c$ on the 2-sphere is rotationally symmetric, we can carry out this method: we simply set $\eta_t(\varphi) = \sqrt{1-2t}$ and $\psi_t(\varphi) = \sqrt{1-2t}\sin\varphi$, we take $\rho_t(\varphi) = \psi_t(\varphi)$. So $\dot\rho_t(\varphi) = \sqrt{1-2t}\cos\varphi$. Since we take φ to range from 0 to π, we use $\pi/2$ as a reasonable lower limit of integration:

$$
\begin{aligned}
(4.2)\quad \zeta_t(\varphi) = \int_{\pi/2}^{\varphi} \sqrt{(1-2t) - (1-2t)\cos^2 u}\, du &= \int_{\pi/2}^{\varphi} \sqrt{1-2t}\sin u\, du \\
&= (\cos\varphi - \cos(\pi/2))\sqrt{1-2t} = \sqrt{1-2t}\cos(\varphi).
\end{aligned}
$$

Plugging in for the formula of a surface of revolution, this gives us the parametrization

$$\Phi_t(\varphi, \theta) = (\sqrt{1-2t}\sin\varphi\cos\theta,\ \sqrt{1-2t}\sin\varphi\sin\theta,\ \sqrt{1-2t}\cos\varphi),$$

exactly as before (it also illustrates exactly where the that very usual parametrization of the 2-sphere comes from, namely, revolving a circle about the z-axis).

4.2 The Rosenau Solution

It's about time we actually tried to visualize a more interesting solution. The **Rosenau solution** on $\mathbb{R} \times S^1(2)$ is given by $g(x, t) = u(x, t)g_c(x)$ where g_c is the usual flat metric on a cylinder of radius 2, namely $dx^2 + 4d\theta^2$, and

$$u(x, t) = \frac{\sinh(-t)}{\cosh(x) + \cosh(t)}$$

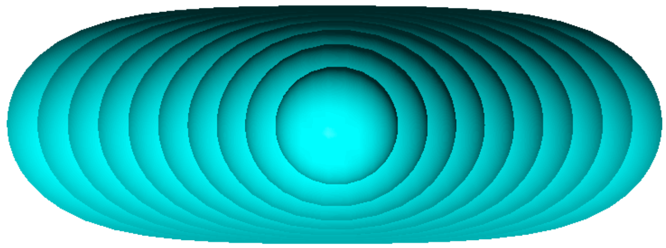

Figure 2: The Rosenau Solution, cutaway view.

It is an *ancient solution* to Ricci Flow, existing on $(-\infty, 0)$. One can verify this using $\mathrm{Rc}(g(t)) = \frac{1}{2}Rg(t)$ in 2 dimensions, and using a formula for conformal changes in metric: given $u \in C^\infty(M)$ with $u > 0$, such that $g = uh$, we have $R_g = \frac{1}{u}(R_h - \Delta_h \log u)$. Here, since $h = g_c$ the cylindrical metric, we have $R_h = 0$ and $\Delta_h = \frac{\partial^2}{\partial x^2}$. After messing around with calculus, we have that

$$R_{g(t)} = \frac{1 + \cosh(x)\cosh(t)}{\sinh(-t)(\cosh(x) + \cosh(t))^2}$$

...

It can be shown that the Rosenau solution extends to a solution on the sphere. Looking at this parametrization we have derived, it shows that the ends of the cylinder close up as $x \to \pm\infty$. Also, as $t \to 0$, the solution looks more and more like a shrinking sphere. It can be shown that the solution tends to a round point. If we had started out with the cylinder with radius 1 instead of 2, we get a different result (not to mention a somewhat adjusted parametrization), which shows, as $x \to \pm\infty$, the surface closing up to cone-points, and thus there is no smooth extension to the sphere (see the other figure). But the existing solution still flattens out very nicely as in the previous case. On the other hand, in the shrinking stage, the shape it tends to is not spherical; it is what I describe as a tamale or *zòng zi*.

Figure 2 shows a similar cutaway view of the (extendible) solution at several times, up to $t = -1/2$, as nested spheres. The larger, oblong spheres correspond to more negative times. As $t \to 0$ the situation essentially becomes that of the nested spheres.

Looking at one end (namely, looking at the flow via dilation about one of the "poles" in the Rosenau solution extended to S^2), as time goes to $-\infty$, it starts to resemble another solution of Ricci flow, called the *cigar soliton*. In fact both these cases, the cigar approaching $-\infty$ and the shrinking sphere approaching 0 are part of an important class of solutions called *Ricci solitons*.

5 Ricci Solitons

Ricci solitons are important because they often occur as limits in various solutions to Ricci flow. Moreover, Ricci solitons themselves define special solutions to RF. In our 2D case, the important examples are all visualizable as surfaces of revolution.

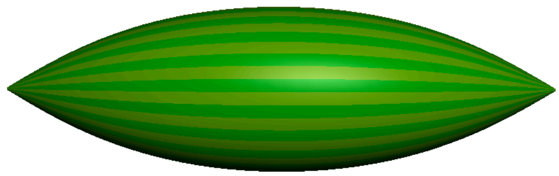

Figure 3: A Rosenau Solution which does not extend to a flow on a sphere.

Recall that a *Ricci soliton* is a Riemannian manifold (M, g) such that $-2\,\mathrm{Rc} = \mathscr{L}_X g + 2\lambda g$ where X is some (time-independent) vector field on M and $\lambda \in \mathbb{R}$. To define a solution of Ricci flow, given a Ricci soliton, we let $\sigma(t) = 1 - 2\lambda t$, $Y(x,t) = \frac{1}{\sigma(t)}X(x)$ and ψ_t the time-dependent flow (the family of diffeomorphisms) generated by Y. Then set $g(t) = \sigma(t)\psi_t^* g$. It can be shown that this gives a solution to Ricci flow (details in [CK]). In fact, given any σ a function defined on some interval containing 0 (without loss of generality we can take $\sigma(0) = 1$), and ψ_t a family of diffeomorphisms, if $g(t) = \sigma(t)\psi_t^* g_0$ for some fixed g_0, we say the solution is *self-similar*. A self-similar solution determines a Ricci soliton at time 0 (we simply differentiate the metric at that time and the usual product rule gives us $\lambda = -\frac{1}{2}\sigma'(0)$ and the vector field to be time-zero field generated by ψ_t, so Ricci solitons and self-similar solutions are equivalent concepts (however the same self-similar solution can be represented by different solitons; taking a derivative at time 0 merely selects a "canonical representative").

5.1 Geometric Realization Revisited

The special case of $\sigma \equiv 1$, namely evolution of metrics by diffeomorphism-pullback alone, is actually an interesting special case. Suppose $\psi_0 = Id_M$ and $g(t) = \psi_t^* g_0$. Then we have that the diffeomorphisms $\psi_t : (M, g(t)) \to (M, g_0)$ makes $(M, g(t))$ into its own geometric realization! More dramatically, this means, by definition of pullback, that all of $(M, g(t))$ are isometric to $(M, g(0))$ for all t . If we have an additional isometric embedding $\iota : (M, g_0) \hookrightarrow (\mathbb{R}^n, g_e)$, say, then $\iota_t = \iota \circ \psi_t$ form a physical realization of $(M, g(t))$. This is easy to see since $\iota_t^* g_e = (\iota \circ \psi_t)^* g_e = \psi_t^* \iota^* g_e = \psi_t^* g_0 = g(t)$. The situation for pulling metrics back also has a dual viewpoint: either we do the passive "borrowing rulers and protractors" from a different part of M (sliding coordinate charts around on M), or we actively deform M to move everything to the new location. If ψ_t is a continuous family, then such diffeomorphisms amount to isotopy and it thus can be visualized as sliding, stretching, or other deforming the manifold inside of itself to a new location. If we want to get back to the "curve" description, the "native" situation of RF is that the curve is fixed and the coordinate grid slides over it, and the "realized" situation means the curve moves along the manifold via the diffeomorphisms, keeping the metric fixed. It's a little harder to imagine since other realizations we've seen have a physically changing set, while here, the set is being moved into itself so you don't see anything happening without making a mark. Think of a spinning sphere vs. a shrinking sphere.

As for the general self-similar solution, with the factor of $\sigma(t)$, this corresponds to rescaling and diffeomorphism put together. If we can isometrically embed M into some $\mathbb{R}^n$ (it's always possible, via the famous Nash Embedding Theorem), we can consider dilation in $\mathbb{R}^n$ by $\sqrt{\sigma(t)}$, in conjunction with embedding process above, namely we take $\iota_t(p) = \sqrt{\sigma(t)}(\iota \circ \psi_t)$. Then

$$\iota_t^* g_e(v, w) = (\sqrt{\sigma(t)}\iota \circ \psi_t)^* g_e(v, w) = \psi_t^* \iota^* (g_e(\sqrt{\sigma(t)}v, \sqrt{\sigma(t)}w) = \sigma(t)\psi_t^* g(v, w) = g(t)(v, w).$$

The shrinking sphere is an example of this, since it takes all diffeomorphisms to be the identity and $\sigma(t) = 1 - 2t$. As you can see, the embedded manifolds essentially "hold their shape" as the term *self-similar* might suggest. The term *soliton* originated from PDE theory to describe wave phenomena that held their shape (never dissipating) for a long time. This motivates the description of self-similar solutions as being *generalized fixed points* or "fixed points modulo diffeomorphism action" of the Ricci flow in [CK].

5.2 The Cigar Soliton

The *cigar soliton* is the plane $\mathbb{R}^2$ with the metric

$$g_\Sigma = \frac{dx^2 + dy^2}{1 + x^2 + y^2} = \frac{dr^2 + r^2 d\theta^2}{1 + r^2}.$$

With a change of variables $r = \sinh(s)$ we can consider the Cigar as a solution on the plane minus the origin, or a half-cylinder $(0, \infty) \times S^1$ given by the metric

$$g_\Sigma = ds^2 + \tanh^2(s) d\theta^2.$$

The proof that it is indeed a soliton, in fact a steady gradient soliton is given in [CK]. The function is given by $f(s) = 2 \log \cosh(s)$, the gradient field $X(s, \theta) = -\nabla f(s) = -2 \tanh(s) \frac{\partial}{\partial s}$. Geometrically, this says that the surface is being propagated along this vector field pointing in the negative s-direction. The Gauss curvature K is given by $2 \operatorname{sech}^2(s)$ which reaches the maximum 2 at $s = 0$, and decays as e^{-2s} as $s \to \infty$ since $\operatorname{sech}^2(s) = \frac{4}{2+e^{-2s}+e^{2s}}$. What this means is that it becomes more and more like a cylinder, exponentially fast, as $s \to \infty$. A steady soliton means that we can take $\sigma(t) \equiv 1$ and thus the solution to Ricci Flow is simply the pullback of g_Σ via the flow given by this vector field.

Let's look at a geometric realization. This amounts to finding just one isometric embedding into $\mathbb{R}^3$, by the results of the last section. As it turns out, there is a rather nasty integral here (it is solvable in terms of elementary functions, as checked out on *Mathematica*, but we won't concern ourselves too much with the specific nature of the solution). We have the rotationally symmetric case with $\eta \equiv 1$ and $\psi(s) = \tanh(s)$. We take $\rho(s) = \psi(s) = \tanh(s)$ and find $\dot{\rho}(s) = \operatorname{sech}^2(s)$. This means, plugging into our formula, that we must take

$$\zeta(s) = \int_a^s \sqrt{1 - \operatorname{sech}^4(u)}\, du$$

which looks none too pleasant. We see, however, that $0 < \operatorname{sech}(u) \leq 1$ so that the term under the square root is also between 0 and 1 and hence, being continuous, is integrable, defining a nice, increasing function of s, despite the apparent intractability. For convenience we take $a = 0$; the coordinates we are considering only $s > 0$ anyhow. We plot this as revolved about the y-axis rather than the z-axis, using $\Phi(s, \theta) = (\rho(s) \sin(\theta), \zeta(s), \rho(s) \cos(\theta))$ (see Fgure 4).

Plugging in the "degenerate" point $s = 0$, where Φ is no longer an embedding as a map from $[0, \infty) \times (0, 2\pi)$, we get $(0, 0, 0)$. But, because the tangent to the generating curve is horizontal (namely,

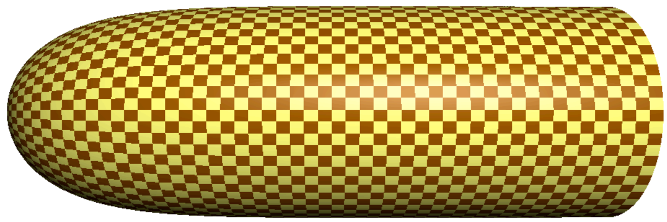

Figure 4: Cigar Soliton

the derivative of the generating curve $(\rho'(s), \zeta'(s)) = (1, 0)$, the parametrized surface (which will be our embedding of the cigar) is smooth at that point, so the degeneracy of the metric there is just due to the usual polar coordinate singularity at 0, and nothing especially geometrically interesting is going on there (well, it is a point of maximum curvature).

As remarked before, dilating about either end of the Rosenau solution yields something more and more cigar-like as we go back in time. In fact this gives kind of an cool picture of convergence of manifolds in general. Dilating about one end amounts to "running alongside" the evolving solution; as long as we "keep up" with the moving end, we will witness it becoming more and more cigar-like. On the other hand, if we stick around in the middle of this realization, the only thing we see is convergence to an infinite cylinder back in time. The situation is analogous to the moving characteristic function $f_t = \chi_{(t,t+1)} \to 0$ "losing the mass at $t = \infty$" since $\int f_t = 1$, so the interchange of limits and integration doesn't work. Similarly, here we "lose curvature" at infinity if we are always picking points corresponding to the embedded points in the xz-plane.

Another (much easier to visualize) soliton is simply taking the vector field $X(x, y) = (x, y)$ in the plane, the Euclidean metric, and $\sigma(t) = 1 + 2t$. Here $Y(t, x, y) = \frac{(x,y)}{1+2t}$, yielding the time-dependent flow $\psi_t(x, y) = \sqrt{1 + 2t}(x, y)$.

This is an expanding soliton; if we consider just the pullback via the flow without regard the scaling, it "stretches the fabric" of the plane more and more. If we look at the solution to Ricci flow we note that the effects of the expansion and the flow *cancel out*, so this is a soliton corresponding to the trivial solution of RF, an unchanging metric.

6 Further Remarks

Recall that in the Rosenau solution we made a big fuss about extendibility to the sphere, and why we mysteriously used the cylinder of radius 2 rather than that of 1. The Rosenau solution is so-called because it involves the solution of the PDE $u_t = (\log u)_{xx}$, yielding a solution of the general form

$$u(x, t) = \frac{2\beta \sinh(-\alpha\lambda t)}{\cosh(\alpha x) + \cosh(\alpha\lambda t)}$$

where α, β, γ are some real constants of integration, satisfying $2\beta\lambda = \alpha$. It turns out that we generate a whole family of Rosenau solutions by using these different u's to multiply the original metric g_c. It

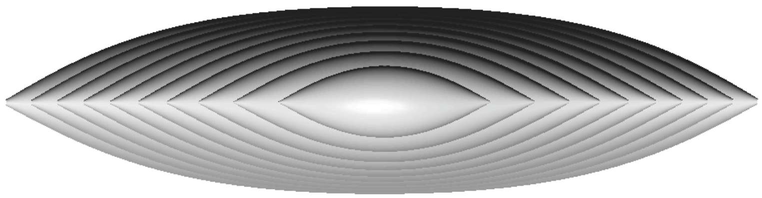

Figure 5: Rosenau "Tamale" Solution with $\alpha < 1$

turns out, when g_c is the cylinder of radius 2, the solution extends to a smooth solution on the sphere if and only if $\alpha = 1$. We might want to ask what exactly happens when α is some other value (we take $\lambda = 1$ for convenience). As the figures demonstrate, the two ends of the cylinder still "collapse," but they collapse to *cone points*---obviously we see the sharp corners (instead of the shrinking sphere at time 0, we have the shrinking *zòng zi*). The general proof is given via the useful "Cylinder-to-Sphere rule" (again, detailed in [CK]), but we might like to get a more direct way of experiencing this, via none other than the famous Gauß-Bonnet theorem. In short, if we integrate the total scalar curvature (twice the Gauss curvature) of the metric ug_c, we will get a result depending on α. Because the Gauß-Bonnet theorem says that the total Gauß curvature is independent of metric, being a topological invariant. Since points have measure zero, therefore, extending the domain of integration to a sphere would not change the integral of the total scalar curvature (the non-smoothly extended metric to the additional two points are irrelevant). It follows that, if the metric really does extend smoothly, the value of α must be that which gives the magic number. It is plainly visible that Rosenau solutions with $\alpha < 1$ are "curvature-deficient" and thus must make up the deficit by having infinite curvature at the ends. See Figure 5.

References

[CK] Bennett Chow and Dan Knopf, *The Ricci Flow: An Introduction*. Providence, RI: American Mathematical Society, 2004.

[Pet] Peter Petersen, *Riemannian Geometry*, 2nd ed. Graduate Texts in Mathematics, **171**. New York: Springer, 2006.