Uniform Continuity

Def. \( f: X \rightarrow Y \), \( X, Y \) metric spaces.

**f is uniformly continuous if**

\[ \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } d_X (x, x') < \delta \Rightarrow d_Y (f(x), f(x')) < \varepsilon, \forall x, x' \in X. \]

Note: \( f \) uniformly continuous on \( X \) \( \Rightarrow \)

\( f \) continuous at every \( x \in X \), but for a given \( \varepsilon > 0 \), the \( \delta \) defining the continuity at \( x \) is independent of \( x \).

Ex 1. \( X = [0,1] \), \( Y = \mathbb{R} \), \( f(x) = x^2 \)

For \( x, x' \in X \), \( |f(x) - f(x')| = |x^2 - x'^2| = |x - x'| |x + x'| \)

since \( x + x' \leq 2 \)

\[ \leq 2 |x - x'| \]

\( f \) is uniformly continuous; for \( \varepsilon > 0 \), take \( \delta = \varepsilon/2 \). Then (\( \star \)) holds.

Note: If \( f: X \rightarrow Y \) continuous at every point \( x \in X \), it need not be uniformly continuous.
Ex 2. \( X = Y = \mathbb{R} \), \( f(x) = x^2 \). \( f \) is continuous i.e. continuous at every \( x \in X \). However claim: \( f \) is not uniformly continuous.

For this, take \( \varepsilon = 1 \), and any \( \delta > 0 \). Then for \( x_1 = \frac{1}{\delta} + \frac{\delta}{2} \), \( x_2 = \frac{1}{\delta} \), \( x_1 - x_2 = \frac{\delta}{2} \), \( x_1 + x_2 \geq \frac{2}{\delta} \).

\[
|f(x_1) - f(x_2)| = (x_1 + x_2)(x_1 - x_2) > \frac{\delta}{2} \cdot \frac{\delta}{2} = 1.
\]

Hence, for \( \varepsilon = 1 \), \( \exists \delta > 0 \) satisfying (**) since \( |x_1 - x_2| = \frac{\delta}{2} < \delta \) \& \( |f(x_1) - f(x_2)| > 1 \).

Thm. \( X \text{ c.p.t.}, f : X \rightarrow Y \) continuous \( \implies f \text{ uniformly continuous.} \)

Pf. (Tricky!). Must show given \( \varepsilon > 0 \) \( \exists \delta > 0 \) s.t.

\[
d_X(p, q) < \delta \implies d_Y(f(p), f(q)) < \varepsilon
\]

For \( p \in X \) by continuity at \( p \), \( \exists \delta_p \) s.t.

\[
d_X(p, q) < \delta_p \implies d_Y(f(p), f(q)) < \frac{\varepsilon}{2}, \quad \forall q \in X.
\]

Since \( X = \bigcup_{p \in X} N_{\delta_p/2} (p) \) \( X \text{ c.p.t.} \implies \exists p_1, \ldots, p_n \in X \)

s.t. \( X = N_{\delta_{p_1}/2} (p_1) \cup \ldots \cup N_{\delta_{p_n}/2} (p_n) \).
We claim \( \delta = \min_{1 \leq i \leq m} \frac{\delta p_i}{2} \) will work, i.e. 

\[
\forall p,q \in X, \quad d_X(p,q) < \delta \implies d_Y(f(p),f(q)) < \varepsilon
\]

Given \( p \in X \), \( \exists j \) s.t. \( d_X(p,p_j) < \frac{\delta p_j}{2} < \delta p_j \)

and hence \((1)\) \( d_Y(f(p),f(p_j)) < \varepsilon/2 \).

\[
\forall q \in X, \quad d_X(q,p_j) \leq d_X(q,p) + d_X(p,p_j)
\]

If \( d_X(p,q) < \delta \) then \( d_X(q,p_j) < \delta + \frac{\delta p_j}{2} \leq \delta p_j \)

by the choice of \( \delta \). Hence

\[
(2) \quad d_Y(f(q),f(p_j)) < \varepsilon/2
\]

We have \( d_Y(f(p),f(q)) \leq d_Y(f(p),f(p_j)) + d_Y(f(p_j),f(q)) \)

by \((1)\) \& \((2)\) \(< \varepsilon/2 + \varepsilon/2 = \varepsilon \). \(\square\)

**Continuity & Connectedness**

Thm. If \( f : X \to Y \) continuous & \( X \) connected,

then \( f(X) \) is connected.

Recall that a metric space \( X \) is connected \(\iff\) the only subsets of \( X \) that are both open & closed are \( X \) and \( \emptyset \).
Cor. (Intermediate Value Theorem)

Let \( f : [a, b] \rightarrow \mathbb{R} \) continuous. If \( f(a) < f(b) \) then for any \( c \), \( f(a) < c < f(b) \), \( \exists x \in (a, b) \) s.t. \( f(x) = c \).

Note: By replacing \( f \) by \(-f\), we obtain an analogous result if \( f(b) < f(a) \).

Pf of Cor 1. Since \([a, b]\) is connected, by Thm, the image \( f([a,b]) \) is also connected.

Since the only connected subsets of \( \mathbb{R} \) are intervals, if \( f(a) < c < f(b) \) then \( c \in f([a,b]) \). //

Cor 2. If \( f : [a, b] \rightarrow \mathbb{R} \) continuous, then \( f([a,b]) = [\inf_{x \in [a,b]} f(x), \sup_{x \in [a,b]} f(x)] \).

Pf. of Cor 2. Since \( f([a,b]) \) is cplt and connected, it must be a closed, bounded interval. Since \( \sup f([a,b]) \) & \( \inf f([a,b]) \) are in \( f([a,b]) \), they must be the endpoints. //