A Proof of Tychonoff’s Theorem

08.11.10

Theorem (Tychonoff). If \((X_\alpha, \tau_\alpha)\) are compact topological spaces for each \(\alpha \in A\), then so is \(X = \prod_{\alpha \in A} X_\alpha\) (endowed with the product topology).

We will prove this theorem using two lemmas, one of which is known as Alexander’s Subbase Theorem (the proof of which requires the use of Zorn’s Lemma).

Lemma 1. Assume \(X\) and \((X_\alpha, \tau_\alpha)\) are as above. Then any open cover of \(X\) consisting solely of elements of the form \(\pi_\alpha^{-1}(O)\) \((O \in \tau_\alpha)\) contains a finite subcover of \(X\).

Proof. Let \(\mathcal{U}\) be such a cover, and define
\[
\mathcal{U}_\alpha = \{O \in \tau_\alpha \mid \pi_\alpha^{-1}(O) \in \mathcal{U}\}
\]
We claim that there is at least one \(\alpha \in A\) such that \(\mathcal{U}_\alpha\) covers \(X_\alpha\): if not, then for each \(\alpha \in A\), there is some \(x_\alpha \in X_\alpha\) such that \(x_\alpha\) is not in the union of all the elements in \(\mathcal{U}_\alpha\). Now define \(f \in X\) via \(f(\alpha) = x_\alpha\). Then \(f\) would not be contained in any of the members of \(\mathcal{U}\), a contradiction since \(\mathcal{U}\) is a cover of \(X\). So choose \(\alpha\) such that \(\mathcal{U}_\alpha\) is a cover of \(X_\alpha\). By compactness, there are \(O_1, \ldots, O_n \in \mathcal{U}_\alpha\) such that \(X_\alpha \subset \bigcup_{j=1}^n O_j\). Then a finite cover of \(X\) is given by \(\{\pi_\alpha^{-1}(O_1), \ldots, \pi_\alpha^{-1}(O_n)\}\).

Now we prove the Alexander Subbase Theorem.

Lemma (Alexander’s Subbase Theorem). Let \((X, \tau)\) be a topological space and \(\mathcal{E}\) be a subbase for \(\tau\). If every collection of sets from \(\mathcal{E}\) that covers \(X\) has a finite subcover, then \(X\) is compact.

Proof. The proof is by contradiction. Suppose every cover of \(X\) by sets in \(\mathcal{E}\) has a finite subcover and \(X\) is not compact. Then the collection \(\mathcal{F}\) of all open covers of \(X\) with no finite subcover is nonempty and partially ordered by set inclusion. With an eye towards Zorn’s Lemma, take any totally ordered subset \(\{E_\alpha\}\) in \(\mathcal{F}\). Then we claim \(E = \bigcup_\alpha E_\alpha\) is an upper bound. To see that \(E\) contains no finite subcover, look at any finite subcollection \(O_1, \ldots, O_n\). Then \(O_j \in E_{\alpha_j}\) for some \(\alpha_j\) (each \(j\)). Since we have a total ordering, there is some \(E_{\alpha_0}\) that contains all of the \(O_j\). Thus, this finite subcollection cannot cover \(X\).
Now Zorn’s Lemma gives us a maximal element $M$ of $\mathcal{F}$. Consider the set $S = M \cap \mathcal{E}$. We claim that $S$ is a cover of $X$. If not, we can find some $x \in X$ that is not in any of the members of $S$. Since $M$ does cover $X$, there is some $O \in M$ with $x \in O$. Since $\mathcal{E}$ is a subbase, there are $V_1, \ldots, V_n$ in $\mathcal{E}$ with $x \in \bigcap^n V_j \subset O$. None of these $V_j$ are in $M$ because then $x$ would be an element of some member of $S$. By maximality of $M$, each $M \cup \{V_j\}$ must contain a finite subcover of $X$, say $X = V_j \cup U_j$, where $U_j$ is a finite union of sets in $M$. Then
\[
O \cup \bigcup^n U_j \supseteq \bigcap^n V_j \cup \bigcup^n U_j \supseteq \bigcap^n (V_j \cup U_j) \supseteq X
\]
This is impossible by construction of $M$. Then $S$ is a cover of $X$. Then because $S$ is contained in $\mathcal{E}$, it would thus have a finite subcover by assumption. This is a contradiction however because $S$ is contained in $M$. Therefore, our original collection $\mathcal{F}$ must be empty so that $X$ is compact. \qed

**Proof of the Main Theorem.**

Take as a subbase for the product topology on $X$ the collection
\[
\{\pi^{-1}_\alpha(O) \mid \alpha \in A, O \in \tau_\alpha}\}
\]
Any subcollection of this set that covers $X$ has a finite subcover by Lemma 1. Thus by Alexander’s Subbase Theorem, $X$ is compact. \qed