A Proof of Tychonoff's Theorem

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Theorem (Tychonoff)**.** *If* (X_α, τ_α) *are compact topological spaces for each* $\alpha \in A$*, then so is* $X = \prod_{\alpha \in A} X_\alpha$ *(endowed with the product topology).*

We will prove this theorem using two lemmas, one of which is known as Alexander's Subbase Theorem (the proof of which requires the use of Zorn's Lemma).

Lemma 1. *Assume* X *and* (X_α, τ_α) *are as above. Then any open cover of* X *consisting solely of elements of the form* $\pi_\alpha^{-1}(O)$ $(O \in \tau_\alpha)$ *contains a finite subcover of* X*.*

Proof. Let $\mathscr{U}$ be such a cover, and define

$$\mathcal{U}_\alpha = \{O \in \tau_\alpha \mid \pi_\alpha^{-1}(O) \in \mathscr{U}\}$$

We claim that there is at least one $\alpha \in A$ such that $\mathcal{U}_\alpha$ covers X_α: if not, then for each $\alpha \in A$, there is some $x_\alpha \in X_\alpha$ such that x_α is not in the union of all the elements in $\mathcal{U}_\alpha$. Now define $f \in X$ via $f(\alpha) = x_\alpha$. Then f would not be contained in any of the members of $\mathscr{U}$, a contradiction since $\mathscr{U}$ is a cover of X. So choose α such that $\mathcal{U}_\alpha$ is a cover of X_α. By compactness, there are $O_1, \ldots, O_n \in \mathcal{U}_\alpha$ such that $X_\alpha \subset \cup_1^n O_j$. Then a finite cover of X is given by $\{\pi_\alpha^{-1}(O_1), \ldots, \pi_\alpha^{-1}(O_n)\}$. □

Now we prove the Alexander Subbase Theorem.

Lemma (Alexander's Subbase Theorem)**.** *Let* (X, τ) *be a topological space and* $\mathcal{E}$ *be a subbase for* τ*. If every collection of sets from* $\mathcal{E}$ *that covers* X *has a finite subcover, then* X *is compact.*

Proof. The proof is by contradiction. Suppose every cover of X by sets in $\mathcal{E}$ has a finite subcover and X is not compact. Then the collection $\mathcal{F}$ of all open covers of X with no finite subcover is nonempty and partially ordered by set inclusion. With an eye towards Zorn's Lemma, take any totally ordered subset $\{E_\alpha\}$ in $\mathcal{F}$. Then we claim $E = \cup_\alpha E_\alpha$ is an upper bound. To see that E contains no finite subcover, look at any finite subcollection $O_1, \ldots, O_n$. Then $O_j \in E_{\alpha_j}$ for some α_j (each j). Since we have a total ordering, there is some E_{α_0} that contains all of the O_j. Thus, this finite subcollection cannot cover X.

Now Zorn's Lemma gives us a maximal element $\mathcal{M}$ of $\mathcal{F}$. Consider the set $S = \mathcal{M} \cap \mathcal{E}$. We claim that S is a cover of X. If not, we can find some $x \in X$ that is not in any of the members of S. Since $\mathcal{M}$ does cover X, there is some $O \in \mathcal{M}$ with $x \in O$. Since $\mathcal{E}$ is a subbase, there are $V_1, \ldots, V_n$ in $\mathcal{E}$ with $x \in \cap_1^n V_j \subset O$. None of these V_j are in M because then x would be an element of some member of S. By maximality of $\mathcal{M}$, each $\mathcal{M} \cup \{V_j\}$ must contain a finite subcover of X, say $X = V_j \cup U_j$, where U_j is a finite union of sets in $\mathcal{M}$. Then

$$O \cup (\cup^n U_j) \supseteq (\cap^n V_j) \cup (\cup^n U_j) \supseteq \cap^n (V_j \cup U_j) \supseteq X$$

This is impossible by construction of $\mathcal{M}$. Then S is a cover of X. Then because S is contained in $\mathcal{E}$, it would thus have a finite subcover by assumption. This is a contradiction however because S is contained in M. Therefore, our original collection $\mathcal{F}$ must be empty so that X is compact. □

Proof of the Main Theorem.

Take as a subbase for the product topology on X the collection

$$\{\pi_\alpha^{-1}(O) \mid \alpha \in A, O \in \tau_\alpha\}$$

Any subcollection of this set that covers X has a finite subcover by Lemma 1. Thus by Alexander's Subbase Theorem, X is compact. □