Math 20E Homework 1

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Problem 5.3 #11: Find

\[ \int \int_D 1 + xy \, dA \]

for the region \( D \) described in the picture.

Method 1:

\[ \int \int_D 1 + xy \, dA = \int \int_D 1 \, dA + \int \int_D xy \, dA \]

Being the area of the region \( D \), the first integral of RHS equals \( \frac{1}{2}(\pi(\sqrt{2})^2 - \pi) = \frac{1}{2}\pi \).
The second integral is split into integrals over two regions

\[ \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} xy \, dy \, dx - \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} xy \, dy \, dx. \]

Integrating against \( y \), the first integral becomes

\[ \int_{-\sqrt{2}}^{\sqrt{2}} x(\frac{2 - x^2}{2}) \, dx \]

The above integral is indeed 0 because the integrand is an odd function and the domain of integration is symmetric. By the same reasoning, \( \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} xy \, dy \, dx = 0 \). Hence,

\[ \int \int_D 1 + xy \, dA = \frac{\pi}{2} \]
Method 2:

\[
\int\int_D (1 + xy) \, dA = \int_0^{\sqrt{2}} \int_0^{2\pi} (1 + r^2 \cos \theta \sin \theta) \, r \, d\theta \, dr = \text{easy} = \pi/2.
\]

Method 3 (highly NOT recommended):

Split the original integral into integrals over three regions

\[
\int\int_D xy \, dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} xy \, dy \, dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} xy \, dy \, dx + \int_{1}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} xy \, dy \, dx
\]

\[
= \int_{-\sqrt{2}}^{\sqrt{2}} \frac{x(2-x^2)}{2} \, dx + \int_{-1}^{1} \frac{x((2-x^2) - (1-x^2))}{2} \, dx + \int_{1}^{\sqrt{2}} \frac{x(2-x^2)}{2} \, dx
\]

\[
= -\frac{1}{8} + 0 + \frac{1}{8} = 0.
\]

\[
\int\int_D 1 \, dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} 1 \, dy \, dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} 1 \, dy \, dx + \int_{1}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} 1 \, dy \, dx
\]

\[
= \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2} \, dx + \int_{-1}^{1} \sqrt{2-x^2} - \sqrt{1-x^2} \, dx + \int_{1}^{\sqrt{2}} \sqrt{1-x^2} \, dx
\]

\[
= \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2} \, dx - \int_{-1}^{1} \sqrt{1-x^2} \, dx
\]

Substituting \( x \) in the first integral with \( \sqrt{2} \sin \theta \) and the second with \( \sin \theta \), we finally get

\[
\int\int_D 1 \, dA = \pi - \frac{\pi}{2} = \frac{\pi}{2}.
\]
Problem 5.3 #15: Find the volume $V$ of the region inside the surface $z = x^2 + y^2$ and between $z = 0$ and $z = 10$.

Method 1:

The volume can be written as

$$\int_{-\sqrt{10}}^{\sqrt{10}} \int_{10-x^2}^{10} 10 - (x^2 + y^2) \, dy \, dx$$

Use polar coordinate to convert the above integral to

$$\int_{0}^{2\pi} \int_{0}^{\sqrt{10}} (10 - r^2) r \, dr \, d\theta = 2\pi \int_{0}^{\sqrt{10}} (r - r^3) \, dr = 50\pi.$$

Method 2:

By the Slice Method-Cavalieri’s Principle in Section 5.1, the volume $V$ is

$$\int_{0}^{10} \pi (\sqrt{z})^2 \, dz = 50\pi.$$