1, 2, 4, 5, 6, 8, 11, 13, 20

1. If $r+x=q$ is rational then $q-r=x$ is rational $\rightarrow\leftarrow$.

If $rx=q$ is rational then $x=q/r$ is rational $\rightarrow\leftarrow$.

2. $\sqrt{12}=2\sqrt{3}$ so by #1 WTS $\sqrt{3}$ is irrational.

If $3=p^2/q^2 \Rightarrow p^2=3q^2 \Rightarrow 3|p^2$

$\Rightarrow 3|p \Rightarrow 9/p^2 \Rightarrow 3|q^2$

$\Rightarrow p, q$ have a common factor $\rightarrow\leftarrow$.

4. Let $x \in E$. Then $\alpha \le x \le \beta$.

5. Let $c=\inf A$. WTS $c=-\sup -A$

or $-c=\sup -A$.

① $-c$ an upper bound of $-A$?
c is upper bound of A so $-c$ is upp bd of $-A$.

② if $\gamma<-c \Rightarrow \gamma$ not an upp bd of $-A$?
$-\gamma$ not an upp bd of A so $-\gamma$ not upp bd of $-A$.

6. a) $((b^m)^{1/n})^{np} = b^{mp} = (b^p)^m = ((b^p)^{1/q})^{mq}$

Since $np = mq$ $\quad (b^m)^{1/n} = (b^p)^{1/q}$ by 1.21.

b) Let $r = m_1/n_1 \quad s = m_2/n_2$

$$b^{r+s} = b^{m_1/n_1 + m_2/n_2} = b^{\frac{m_1 n_2 + m_2 n_1}{n_1 n_2}}$$

$$= \left(b^{m_1 n_2 + m_2 n_1}\right)^{1/n_1 n_2}$$

$$= \left(b^{m_1 n_2}\right)^{1/n_1 n_2} \cdot \left(b^{m_2 n_1}\right)^{1/n_1 n_2}$$

$$= b^{m_1/n_1} \cdot b^{m_2/n_2} = b^r b^s$$

c) Show b^r is an upper bound and if $\gamma < b^r$ then γ not an upper bound.

- b^r upp bound is trivial.
- if $\gamma = b^t < b^r$ then $t < r$ since $b > 1$ so b^t not an upper bound.

d) Show $\sup B(x+y) = \sup B(x) \cdot \sup B(y)$.

8. If $\mathbb{C}$ is ordered then $\forall x \in \mathbb{C}$

$x^2 > 0$ but $i^2 = -1 < 0$

11. Let $r = |z|$ $w = \frac{z}{|z|}$. Then $z = rw$.

Not unique if $r = 0$.

13. $|x - y + y| \le |x - y| + |y| \Rightarrow |x| - |y| \le |x - y|$

Likewise $|y - x + x| \le |x - y| + |x| \Rightarrow |y| - |x| \le |x - y|$

So $||x| - |y|| \le |x - y|$

16, 18. 2, 3, 4, 5, 6, 11

~~18.~~

18. let $\underline{x} = (x_1, .., x_n)$. ~~If $\exists x_i =$~~

If $\exists i$ s.t. $x_i = 0$ take $y = (0, 0, \overset{i\text{th}}{1}, 0, .., 0)$

then $x \cdot y = 0$.

If all $x_i \neq 0$ Take $y = (x_2, -x_1, 0, .., 0)$

~~2. let $A_{n,N}$ be~~

2. let $A_{n,N} = \left\{ z \in \mathbb{C} \;\middle|\; \begin{array}{l} z \text{ satisfies an algebraic equation} \\ \text{of degree } n \text{ with} \\ n + |a_0| + \cdots + |a_n| = N \end{array} \right\}$.

Claim $A_{n,N}$ is finite — since $a_i \in \mathbb{Z}$ there are finitely many equations ~~with~~ of degree n s.t. $n + |a_0| + \cdots + |a_n| = N$. By Fundamental Thm of Algebra, each equation has finitely many roots $\Rightarrow$ $A_{n,N}$ is finite.

Note: $\{\text{Algebraic \#s}\} = \bigcup_{n,N \in \mathbb{N}} A_{n,N}$.

5

3. The algebraic numbers are countable but the real numbers are uncountable $\Rightarrow$ $\exists$ real but not algebraic numbers.

4. No. Let I = set of irrational numbers. $\mathbb{Q}$ is countable and $\mathbb{R} = \mathbb{Q} \cup I$ so I must be uncountable

5. Take $\{1/n\}_1^\infty \cup \{1 + 1/n\}_1^\infty \cup \{2 + 1/n\}_1^\infty = S$

Limit points are $\{0, 1, 2\}$. To check this show $\forall \varepsilon > 0$ $\exists q \in S$ $q \neq 0$ s.t. $q \in B(0, \varepsilon) \cap S$. Repeat for $1, 2$.

To show $x \in \mathbb{R} \setminus \{0,1,2\}$ is not a limit point

First break $\mathbb{R} - \{0,1,2\}$ into intervals

$(-\infty, 0) \cup \{(1/(n+1), 1/n)\}_1^\infty \cup \{(1 + 1/(n+1), 1 + 1/n)\}_1^\infty \cup \{(2 + 1/(n+1), 2 + 1/n)\}_1^\infty \cup [3, \infty)$

x must be in one of these. Suppose the endpoints of that interval are a, b

Take $\varepsilon = \min\left(\left|\frac{x-a}{2}\right|, \left|\frac{x-b}{2}\right|\right)$. Then $B(x, \varepsilon) \cap S$ is empty.

6. WTS E' has all its limit points.

Let x be a limit point of E'.

<u>Claim</u> x is a limit point of E (Hence $x \in E'$).

Let $\varepsilon > 0$ wts $\exists\, q \in B(x, \varepsilon) \cap E$ with $q \neq x$.

But x a lim pt of E' $\Rightarrow$ $\exists\, q' \in B(x, \varepsilon) \cap E'$ with $q' \neq x$. q' is a limit point of E since $q' \in E'$.

Let $\varepsilon' = |q' - x|/2$. Then $\exists\, q \in B(q', \varepsilon') \cap E$ $q \neq q'$.

But $B(q', \varepsilon') \subset B(x, \varepsilon)$ so $q \in B(x, \varepsilon) \cap E$ and $q \neq x$ (since $x \notin B(q', \varepsilon')$)

$E' = \bar{E}'$? If $x \in E'$ then $x \in \bar{E}'$ since $E \subseteq \bar{E}$. Let $x \in \bar{E}'$ wts x is a limit point of E. Let $\varepsilon > 0$. wts $\exists\, q \in B(x, \varepsilon) \cap E$ with $q \neq x$. Use same logic as above to get result.

$E' = (E')'$? No. Let $E = \{1/n\}_1^\infty$.

Then $E' = \{0\}$ but $(E')' = \emptyset$.

11.

a) No. Let $x=0$, $y=1$, $z=2$.

Then $(x-z)^2 \le (x-y)^2 + (y-z)^2$

$$\underbrace{(x-z)^2}_{=\,4} \not\le \underbrace{(x-y)^2}_{=\,1} + \underbrace{(y-z)^2}_{=\,1}$$

$$4 \not\le 1 + 1$$

b) Yes. $|x-z| \le |x-y| + |y-z|$

$$\Rightarrow \sqrt{|x-z|} \le \sqrt{|x-y|+|y-z|} \le \sqrt{|x-y|} + \sqrt{|y-z|}$$

c) No $d(1,-1)=0$.

d) No. $d(1, 1/2) = 0$.

e) Yes.

7, 8, 9, 10, 12

7 a) $\overline{B_n} \overset{?}{\subseteq} \bigcup_1^n \overline{A_i}$

$B_n \subseteq \bigcup A_i \subseteq \bigcup \overline{A_i}$ But $\bigcup_1^n \overline{A_i}$ is closed

so $\overline{B_n} \subseteq \bigcup_1^n \overline{A_i}$. (Note finiteness is required here).

$\bigcup \overline{A_i} \subseteq \overline{B_n}$?

$A_i \subseteq B_n \subseteq \overline{B_n} \Rightarrow \overline{A_i} \subseteq \overline{B_n} \Rightarrow \bigcup \overline{A_i} \subseteq \overline{B_n}$.

b). Use same logic as (a).

Ex Take $A_i = \{1/i\}$ Then $\bigcup_1^\infty A_i = \{1/n\}_1^\infty$

But $\overline{\bigcup_1^\infty A_i} = 0 \cup \{1/n\}_1^\infty$.

8. If E is open $\overset{?}{\Rightarrow}$ $x \in E$ is a limit point.

Fix $\varepsilon > 0$. Then $\exists\, \varepsilon' < \varepsilon$ s.t. $B(x, \varepsilon') \subseteq E$

But $B(x, \varepsilon') \subseteq B(x, \varepsilon)$ so $\exists\, q \in B(x, \varepsilon) \cap E$ $q \neq x$.

False for closed sets. Take $E = \{(0,0)\}$.

E has no limit points so $(0,0)$ is not a limit point.

9. a) Every point in E° is an interior point of E°

so E° is open.

b) E open $\Rightarrow$ $E^\circ = E$ is trivial by 2.18f.

$E^\circ = E$ $\Rightarrow$ E open is trivial by (a).

c) If $x \in G$ $\exists\, \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq G$.

$\Rightarrow$ $B(x, \varepsilon) \subseteq E$ so $x \in E^\circ$.

d) $(E^\circ)^c \overset{?}{=} \overline{(E^c)}$

Let $x \in (E^\circ)^c$ $\Rightarrow$ x not interior in E. If $x \notin E$ $\Rightarrow$ $x \in E^c$ $\Rightarrow$ $x \in \overline{E^c}$. If $x \in E$, $\forall\, \varepsilon > 0$ $B(x, \varepsilon) \nsubseteq E$ so $\exists\, q \neq x$ s.t. $q \in B(x, \varepsilon) \cap E^c$ $\Rightarrow$ $x \in \overline{(E^c)}$.

Let $x \in \overline{(E^c)}$ If $x \in E^c$ $\Rightarrow$ $x \notin E$ $\Rightarrow$ $x \notin E^\circ$.
If $x \notin E^c$ then $\forall\, \varepsilon > 0$ $\exists\, q \neq x$ s.t. $q \in B(x, \varepsilon) \cap E^c$
$\Rightarrow$ $B(x, \varepsilon) \nsubseteq E$ $\Rightarrow$ $x \notin E^\circ$ $\Rightarrow$ $x \in (E^\circ)^c$.

9e) No. Let $\Sigma = \mathbb{Q}$. Then $\Sigma^\circ = \emptyset$
but $\overline{\Sigma}^\circ = \mathbb{R}$.

f) No. Let $\Sigma = \mathbb{Q}$. $\overline{\Sigma} = \mathbb{R}$ but $\overline{\Sigma^\circ} = \overline{\emptyset} = \emptyset$.

10. This is easily shown to be a metric.

<u>Claim</u> $\{p\}$ is an open set. ($\{p\}$ = singleton point).

But $\{p\} = B(p, 1/2)$ so done.

Hence arbitrary sets S are open since $S = \bigcup_{p \in S} \{p\}$
& open sets are closed under arbitrary union.

Arbitrary sets are closed too since S^c would be open.

Finite sets are compact: If $S = \{p_1, \ldots, p_n\}$. Let $\mathcal{U} = \{U_\alpha\}$
be an open cover. Then $p_i \in U_i$ for some $U_i \in \{U_\alpha\}$.
~~Take~~ Take $U_1 \cup \ldots \cup U_n$ as the finite subcover.

Infinite sets are not compact. We want an open cover
that cannot be refined to a finite subcover.
Take $\mathcal{U} = \{\{p\}\}_{p \in S}$. If we took a finite subcover
then it would have finitely many elements hence
couldn't cover S.

12. Let $\{U_\alpha\}$ be an open cover. 11

Then $0 \in U_0$ for some $U_0 \in \{U_\alpha\}$.

U_0 open $\Rightarrow \exists\, \varepsilon > 0$ s.t. $B(0,\varepsilon) \subseteq U_0$.

Hence $\exists\, N$ s.t. $n \geq N \Rightarrow \frac{1}{n} < \varepsilon$

$\Rightarrow \{0\} \cup \{1/n\}_N^\infty \subseteq U_0$.

Now find U_i s.t. $1/i \in U_i$

for $i = 1, \dots, N$. Then $U_0 \cup \dots \cup U_N$ is a finite open

cover

16 19 22 1 2 3 12

16. $\mathbb{Q}$ has the same metric as $\mathbb{R}$ so it can be considered with the subspace topology.

$E = \{(-\sqrt{3}, -\sqrt{2}) \cap \mathbb{Q}\} \cup \{(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}.\}$.

E is closed since $E^c = \{(-\infty, -\sqrt{3}] \cup [-\sqrt{2}, \sqrt{2}] \cup [\sqrt{3}, \infty)\} \cap \mathbb{Q}$ is open. E is bounded since $E \subseteq B(0,5)$.

E is not compact since by Thm 2.33 $E \subseteq \mathbb{Q} \subseteq \mathbb{R}$ is compact iff $E \subseteq \mathbb{R}$ is compact. But E is not closed in $\mathbb{R}$ since it does not contain all its limit points (e.g. $\sqrt{3}$).

E is open by Thm 2.30

13.

14. a) WTS $\bar{A} \cap B = \emptyset$ & $A \cap \bar{B} = \emptyset$

But $\bar{A} \cap B \subseteq \bar{A} \cap \bar{B} = \emptyset$. Likewise for $A \cap \bar{B} = \emptyset$.

b) Spse $\exists\, p \in A \cap \bar{B}$. Then $\exists\, \varepsilon > 0$ s.t.

$B(p, \varepsilon) \subseteq A$ since A open. Since $p \in \bar{B} \Rightarrow$

if $p \in B$ then $p \in A \cap B$ $\rightarrow\leftarrow$.

If $p \in B'$ then $\exists\, q \in B(p, \varepsilon) \cap B$ $q \neq p$ so

$q \in A \cap B$ $\rightarrow\leftarrow$.

c). $A = B(p, \delta)$ and $B = (B(p,\delta)^c)^\circ$

These are disjoint & open so use (b).

~~d) WLOG assume d~~

d) Let $p, q \in X$. WLOG take $d(p,q) = 1$.

$\forall\, \delta \in (0,1)$ Let $A_\delta = B(p, \delta)$. $B_\delta = (B(p,\delta)^c)^\circ$.

Let $\Sigma_\delta = \{x \mid d(x,p) = \delta\}$. Claim: $\Sigma_\delta \neq \emptyset$.

If $\Sigma_\delta = \emptyset$ Then $X = A_\delta \cup B_\delta \cup \Sigma_\delta = A_\delta \cup B_\delta$

a disconnection. $\rightarrow\leftarrow$.

So let $x_\delta \in \Sigma_\delta$. Do this over all $\delta \in (0,1)$

to get an uncountable set.

22. Claim: $\mathbb{Q}^n \subseteq \mathbb{R}^n$ is a countable dense set.

~~$\mathbb{Q}^n$ ... $\mathbb{Q}^n$ is countable?~~

~~Indeed: $\mathbb{Q}$ is countable. $\mathbb{Q}^{n-1} \times \mathbb{Q}$ ...~~

$\mathbb{Q}^n$ is countable by Thm 2.13.

$\mathbb{Q}^n$ dense? WTS $\forall x \in \mathbb{R}^n, \forall \varepsilon \quad \exists q \in B(x, \varepsilon) \cap \mathbb{Q}^n$.

Let $x = (x_1, \ldots, x_n)$ [struck out]. For $\sqrt{\varepsilon/n}$ $\exists q_i$

s.t. $q_i \in B(x_i, \sqrt{\varepsilon/n}) \subseteq \mathbb{R}^1$. e.g. $x_i - \sqrt{\varepsilon/n} < q_i < x_i + \sqrt{\varepsilon/n}$

so take $q = (q_1, \ldots, q_n)$. Then

$$|x-q| = \sqrt{(x_1-q_1)^2 + \cdots + (x_n-q_n)^2} < \varepsilon.$$

1. <u>Claim</u> $|s_n| \longrightarrow |s|$.

WTS $\forall \varepsilon > 0 \; \exists \; N$ s.t. $n \geq N \Rightarrow \big||s_n| - |s|\big| < \varepsilon$.

But $\big||s_n| - |s|\big| \leq |s_n - s|$ & $\exists \; N$ s.t. $|s_n - s| < \varepsilon$
$\forall \; n \geq N$.

2. $$\lim \sqrt{n^2+n} - n = \lim \sqrt{n^2+n} - n \cdot \left(\frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n}\right)$$

$$= \lim \frac{n}{\sqrt{n^2+n} + n}$$

$$= \lim \frac{1}{\sqrt{1+1/n} + 1}$$

$$= 1/2.$$

3. WTS s_n is bounded & monotone.

s_n monotone? Use induction. $\sqrt{2} = s_1 \leq \sqrt{2+\sqrt{2}} = s_2$

$s_k \leq s_{k+1}$? $\underbrace{\sqrt{2+\sqrt{s_{k-1}}}}_{s_k} \leq \underbrace{\sqrt{2+\sqrt{s_k}}}_{s_{k+1}}$ since $s_{k-1} \leq s_k$.

s_n bounded by 2? Induct. $s_1 = \sqrt{2} < 2$.

$s_k = \sqrt{2+\sqrt{s_{k-1}}} < \sqrt{2+\sqrt{2}} < \sqrt{4} = 2$.

4 5 6 16 20 21 23

4. Induct to show $S_{2n} = \frac{2^{n-1}-1}{2^n}$ $S_{2n+1} = \frac{2^n-1}{2^n}$

Then $S_{2n} \to 1/2$ $S_{2n+1} \to 1$.

Since $S_i < 1$ $\forall i$ 1 is an upper limit.

$\liminf s_i = \lim_{N\to\infty} (\inf_{n \geq N} s_i) = \lim_{N\to\infty} \frac{2^{\frac{N}{2}-1}-1}{2^{N/2}} = 1/2.$

if N is even

or $= \lim_{N\to\infty} \frac{2^{\frac{N-1}{2}}-1}{2^{\frac{N-1}{2}}} = 1/2$

if N is odd.

5. WTS $\sup_{N\leq n}(a_n+b_n) \leq \sup_{N\leq n}(a_n) + \sup_{N\leq n}(b_n)$ $\forall N$.

but if A is an upper bound for $\{a_n\}_N^\infty$ & B is an upper bound for $\{b_n\}_N^\infty$ then $A+B$ is an upper bound for $\{a_n+b_n\}_N^\infty \Rightarrow \sup(a_n+b_n) \leq A+B$

$\Rightarrow \sup(a_n+b_n) \leq \sup(a_n) + \sup(b_n)$

Now note $\limsup a_n = \lim_{N\to\infty} \sup_{N\geq n} a_n$

& take limits in the first line.

6. e) $\sum_{0}^{N} a_n = (\sqrt{1}-\sqrt{0}) + \cdots + (\sqrt{N+1}-\sqrt{N})$
$= \sqrt{N+1}$

$\longrightarrow \infty$ so diverges.

f) $\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{n(2\sqrt{n})} = \frac{1}{2}\cdot\frac{1}{n^{3/2}}$

But $\sum_1 1/n^{3/2}$ converges so $\sum_1 \frac{\sqrt{n+1}-\sqrt{n}}{n}$

converges too.

c) By root test $\limsup \sqrt[n]{(\sqrt[n]{n}-1)^n} = \limsup \sqrt[n]{n}-1$
$= 0 < 1$

So converges.

~~[illegible]~~

20 Let $\varepsilon > 0$ WTS $\exists N$ s.t. $|p_n - p| < \varepsilon$

But $\exists N_1$ s.t. $n_k \geq N_1 \Rightarrow |p_{n_k} - p| < \varepsilon/2$

$\exists N_2$ s.t. $m, n \geq N_2 \Rightarrow |p_m - p_n| < \varepsilon/2$

So $|p_n - p| \leq |p_n - p_{n_k}| + |p_{n_k} - p| < \varepsilon$

for ~~the~~ $n \geq \max(N_1, N_2) = N$.

21. <u>Claim</u> $\bigcap E_n \neq \emptyset$. Let $x_i \in E_i$

~~$\exists$~~ take the sequence $\{x_i\}$. $\forall \varepsilon > 0$ $\exists N$ s.t.

$|x_m - x_n| < \varepsilon$ since $\exists N$ s.t. $\operatorname{diam} E_n < \varepsilon$

for $n \geq N$.

X complete $\Rightarrow$ $x_n \longrightarrow x \in X$. $\Rightarrow$ $\bigcap E_n \neq \emptyset$.

If $\bigcap E_n$ has more than one point then $\exists p, q \in \bigcap E_n$

and $\operatorname{diam} E_n \geq d(p,q) > 0$ meaning $\operatorname{diam} E_n \not\to 0$

$\longrightarrow \Leftarrow$.

23. Let $\varepsilon > 0$ WTS $\exists N$ s.t. $n, m \geq N \Rightarrow$

$$|d(p_n, q_n) - d(p_m, q_m)| < \varepsilon.$$

But $\exists N_p$ s.t. $n, m \geq N_p \Rightarrow d(p_n, p_m) < \varepsilon/2$

$\exists N_q$ s.t. $n, m \geq N_q \Rightarrow d(q_n, q_m) < \varepsilon/2$

So the hint implies

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_n, q_m)$$

Note: interchange p and q in hint to get absolute value.

$\Rightarrow$ for $N = \max(N_p, N_q)$ then

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_n, q_m) < \varepsilon.$$

7 8 9 10 12 13 20.

7. $\sum_1^{\infty} \frac{\sqrt{a_n}}{n} \leq \sqrt{\sum_1^{\infty} a_n \sum_1^{\infty} 1/n^2}$ & RHS converges

so LHS converges.

8. <u>Claim</u> $\sum_1^{\infty} a_n b_n$ converges $\iff \sum_1^{\infty} a_n (b_n - b)$ converges.

~~$\Rightarrow \sum_1^{\infty} a_n(b_n - b) = \sum_1^{\infty} a_n b_n$~~

~~$\Rightarrow \sum_1^{\infty} a_n b_n + \sum$~~

$\underline{\Rightarrow}$ $\sum_1^{N} a_n b_n - \sum_1^{N} b a_n = \sum_1^{N} a_n (b_n - b)$ but LHS

converges so RHS converges.

$\underline{\Leftarrow}$ $\sum_1^{\infty} a_n (b_n - b) + \sum_1^{\infty} a_n b$ converges so

$\sum_1^{\infty} a_n(b_n - b) + a_n b = \sum_1^{\infty} a_n b_n$ converges.

WLOG assume b_n monotonically decreasing. Use

3.42 to show $\sum_1^{\infty} a_n (b_n - b)$ converges

$\Rightarrow \sum_1^{\infty} a_n b_n$ converges.

9. a) $\limsup \sqrt[n]{n^3} = \limsup (\sqrt[n]{n})^3 = 1$

So $R = 1$

b) ~~$\limsup \sqrt[n]{2^n}$~~ $\limsup \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \limsup \frac{2}{n+1} = 0$

So $R = \infty$.

c) ~~$\limsup \frac{2^{n+1}}{n+1}$~~ $\limsup \sqrt[n]{\frac{2^n}{n^2}} = 2 \cdot \limsup \frac{1}{\sqrt[n]{n^2}} = 2$

$R = 1/2$

d) $\limsup \sqrt[n]{n^3/3^n} = \frac{1}{3} \limsup (\sqrt[n]{n})^3 = 1/3$

$R = 3$

~~10. Sup~~

10. $\sup\limits_{n \geq N} \sqrt[n]{a_n} > 0$ since $\exists$ infinitely many non zero integers in the sequence. Moreover.

$\sup\limits_{n \geq N} \sqrt[n]{a_n} \geq 1$ since if $a_n \in \mathbb{Z} - \{0\}$

then $\sqrt[n]{a_n} \geq 1 \iff |a_n| \geq 1^n = 1$.

Hence $\limsup \sqrt[n]{a_n} \geq 1$.

12. a) If $m < n \Rightarrow r_n < r_m$

So $\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m + \cdots + a_n}{r_m} = \frac{r_m - r_n + a_n}{r_m}$

$> 1 - \frac{r_n}{r_m}$.

By Thm 3.22 take $\varepsilon = 1/2$.

then $\sum\limits_{m}^{n} \frac{a_n}{r_m} > 1 - r_n/r_m \longrightarrow 1$ as $n \to \infty$

So contradiction.

B. WTS $\sum_1^\infty |c_n|$ converges.

~~So WTS~~ But $|c_n| \le \sum_0^n |a_k||b_{n-k}| = d_n$

So WTS $\sum_1^\infty d_n$ converges but this is found by 3.50.

1. No. Take $f = \begin{cases} 1 & x=0 \\ 0 & \text{else} \end{cases}$

$\lim f(x+h) - f(x-h) = 0.$

2. If $x \in E$ then $f(x) \in f(E) \subseteq \overline{f(E)}$.
If $x \in E'$ WTS $f(x)$ is a limit pt of $f(E)$.
Let $B(f(x), \varepsilon) \subseteq Y$. WTS $\exists q \in B(f(x), \varepsilon)$ s.t. $q \neq f(x)$ $q \in f(E)$.
Cont. $\Rightarrow$ $\exists\ \delta$ s.t. $f(B(x,\delta)) \subseteq B(f(x), \varepsilon)$.
$x \in E' \Rightarrow \exists\ p$ s.t. $p \neq x$, $p \in B(x,\delta) \cap E$
then $f(p) \in B(f(x), \varepsilon) \cap f(E)$
If $f(p) \neq f(x)$ $\Rightarrow$ done. If $f(p) = f(x)$ $\Rightarrow$ $f(x) \in f(E) \subseteq \overline{f(E)}$.

Ex: $f: \mathbb{R}^\times \to \mathbb{R}$ by $f(x) = 1/x$. Let $E = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.
So $\overline{E} = \mathbb{R}^\times$ but $f(E) = \mathbb{R} - \{0\}$ yet $\overline{f(E)} = \mathbb{R}$.

3. $f^{-1}(0)$ is closed & $f^{-1}(0) = Z(f)$

4. WTS $\overline{f(E)} = f(X)$. But $f(\overline{E}) = f(X) \subseteq \overline{f(E)}$
Clearly $\overline{f(E)} \subseteq f(X)$ as $\subseteq$ is smallest top of $f(X)$.

WTS $g(p) = f(p)$ on $E \Rightarrow$ $=$ on X.
or $g - f = 0$ on E.

Reduce to if $f = 0$ on $E \Rightarrow f = 0$ on X.
But $f(E) = \{0\}$ so $\overline{f(E)} = \{0\}$ since
Hausdorff. so $f(\overline{E}) = f(X) \subseteq \overline{f(E)} = \{0\}$.

7. f bdd on $\mathbb{R}^2$?

$f = \frac{xy^2}{x^2+y^4}$

NOTE $(x-y^2)^2 \geq 0$

$\Rightarrow x^2+y^4 \geq 2xy^2$

$\Rightarrow 1/2 \geq xy^2/x^2+y^4$

$g = \frac{xy^2}{x^2+y^6}$

g unbdd in every nbhd of $(0,0)$?

WTS $\exists$ $x_i, y_i \to (0,0)$ that becomes unbdd. Take $(1/n^3, 1/n)$

$\Rightarrow g(1/n^3, 1/n) = n \longrightarrow \infty$.

f not cont at $0,0$? want 2 sequences $\to 0$ that converge to diff values

Take $(0, 1/n)$ & $(1/n^2, 1/n)$
$\downarrow \qquad\qquad \downarrow$
$0 \qquad\qquad 1/2$.

Straight lines of form $y = ax$ $a \in \mathbb{R}$.
or $x = 0$.

Restriction is setting $y = ax$ or $x = 0$ & check cont.

8. Let $\varepsilon = 1$. $\exists\, \delta$ s.t. $|p-q| < \delta \Rightarrow |f(p) - f(q)| < 1$.
E bdd $\Rightarrow$ $E \subseteq (-\delta M, \delta M)$ for some $M \in \mathbb{N}$.

Fix $p \in E$.

WTS $|f(x)| \le 2M + |f(p)|$.

WLOG $x < p$ else same logic. $\exists\, N < 2M - 1$ s.t. $p \in (x + N\delta, x + (N+1)\delta)$

$$\begin{aligned}
|f(x)| &\le |f(x) - f(x+\delta)| + |f(x+\delta)|. \\
&< 1 + |f(x+\delta)|. \\
&< N + |f(x + N\delta)| \\
&\le N + |f(x+N\delta) - f(p)| + |f(p)| \\
&< N + 1 + |f(p)|. \\
&\le 2M + |f(p)|.
\end{aligned}$$

14. WTS $\exists\, c$ s.t. $f(c) - id(c) = 0$.

If $f(0) - id(0) = 0$ done so assume $f(0) > 0$
If $f(1) - 1 = 0$ done so assume $f(1) - 1 < 0$

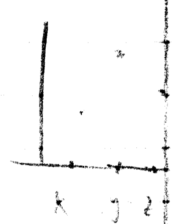
By Thm 4.23 $f(1) - 1 < 0 < f(0)$
$\Rightarrow \exists\, x$ s.t. $f(x) - x = 0$ $\blacksquare$.

15. WTS ① open maps are injective
② Continuous + injective ⇒ monotone.

① Spse $\exists\ x, y \in \mathbb{R}$ st. $f(x) = f(y)$.
WLOG $x<y$. By 4.16 $\exists$
t st. $f(t) = \max\{f(s) \mid s \in [x,y]\}$.

C1. $f(x) < f(t)$ Then $f(t)$ is not interior since $\forall\ \varepsilon$ $B(f(t), \varepsilon) \notin f((x,y))$. $\rightarrow\leftarrow$.

C2. If $f(x) = f(t)$ $\exists\ t'$ st. $f(t') = \min\{f(s) \mid s \in [x,y]\}$
if $(f(x) > f(t'))$ done similarly as above.
if $f(x) = f(t')$
$\Rightarrow f((x,y)) = \{f(x)\} \Rightarrow$ not open



Monotone? WLOG assume $x<y<z$ s.t.
$f(x) < f(y)$ + $f(y) > f(z)$

C1 $f(x) = f(z)$ $\rightarrow\leftarrow$ injective
C2 $f(x) < f(z)$
$\Rightarrow f(x) < f(z) < f(y) \Rightarrow \exists\ t \in [x,y]$
s.t. $f(t) = f(z)$ $\rightarrow\leftarrow$ inject.
C3 $f(z) < f(x)$
$\Rightarrow f(z) < f(x) < f(y)$
$\Rightarrow \exists\ t \in [y,z]$ s.t. $f(t) = f(x)$ $\rightarrow\leftarrow$.