1. (a)

$$f' = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}$$

hence $Jf = -8xy$. So $Jf = 0$ iff $x = 0$ or $y = 0$. Hence outside the coordinate axes, the inverse function theorem applies.

(b)

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0)) \cdot \gamma'(t_0) = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 22 \\ -10 \end{pmatrix}$$

2. $f(x, y) = (x^3 + y^2, x^2 + y^3)$. So

$$f'(x, y) = \begin{pmatrix} 3x^2 & 2y \\ 2x & 3y^2 \end{pmatrix} \qquad f'(1, 1) = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Hence $Jf(1, 1) = 5 \neq 0$ so we can find an inverse by the inverse function theorem.

3. *Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be given by $f(A) = A^2$. Show that near the identity matrix there is a continuous square-root function g—e.g. a map $g : \mathbb{R}^4 \to \mathbb{R}^4$ s.t. $f \circ g(B) = B$.*

I want to use the inverse function theorem on f near the identity matrix.

$$f(A) = A^2 \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2 x^3 & x_1 x_2 + x_2 x_4 \\ x_3 x_1 + x_4 x_3 & x_2 x_3 + x_4^2 \end{pmatrix}.$$

This can be rewritten as

$$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2 x^3, x_1 x_2 + x_2 x_4, x_3 x_1 + x_4 x_3, x_2 x_3 + x_4^2).$$

The derivative is

$$f'(x_1, \ldots, x_4) = \begin{pmatrix} 2x_1 & x_3 & x_2 & 0 \\ x_2 & x_1 + x_4 & 0 & x_2 \\ x_3 & 0 & x_1 + x_4 & x_3 \\ 0 & x_3 & x_2 & 2x_4 \end{pmatrix}$$

so we have

$$f' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = f'(1, 0, 0, 1) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 01 & \\ 0 & 0 & 0 & 2 \end{pmatrix}, \qquad Jf(1, 0, 0, 1) = 4.$$

Hence $Jf \neq 0$ at the identity so there is an inverse by the inverse function theorem.

4. $x^3 + 3x^2 + 2x - 6$ *has a zero at* $x = 1$. *Suppose we change the coefficient of* x *a little, does it change the zero* $x = 1$ *a little as well?*

   Let $f(x, y) = x^3 + 3x^2 + yx - 6$. We want to see if we can get a continuous function $x = g(y)$ near $(1, 2)$. If we can then continuity of g would guarantee that changing y slightly would change x slightly as well.

   But $f' = (3x^2 + 6x + y, x)$ so $f'(1, 2) = (10, 1) = (A_x, A_y)$. The implicit function theorem says we can find such a g.

5. *Let* S *be the set of solutions to* $x^3 + y^3 + z^3 + (x + y + z)^3 = 1$. *Show for any* $p \in S$ *you can find a neighborhood* $U \subset \mathbb{R}^3$*, a neighborhood* $V \in \mathbb{R}^2$*, and a differentiable map* $\varphi : V \to U \cap S$.

   Let $f(x, y, z) = x^3 + y^3 + z^3 + (x + y + z)^3 - 1$; then

   $$f' = (3x^2 + 3(x + y + z)^2, 3y^2 + 3(x + y + z)^2, 3z^2 + 3(x + y + z)^2).$$

   Note $f' = (0, 0, 0)$ iff $x = y = z = 0$, but $(0, 0, 0) \notin S$. Hence for every point of S, one of A_x, A_y, A_z will be nonzero. WLOG assume $A_x \neq 0$. Then you can use the implicit function theorem to find the desired neighborhoods and maps.