1. Let p be a polynomial of degree n. Then the preimage of a general point is n points. In a small nbhd of each of these points p is a local homeomorphism. By Prop 2.30

$\deg p = \sum_i \deg p|_{x_i}$ where x_i is a preimage of the previously chosen point. Since we have a local homeo, $\deg p|_{x_i} = 1 \Rightarrow \deg p = n$.

2. ~~Recall~~

<u>Thm</u> (Fund. Thm of Finitely Gen Abelian Gps).

If G is fin. gen ab gp $\Rightarrow$

$$G \simeq \mathbb{Z}^m \oplus \mathbb{Z}/p_1^{d_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k^{d_k}\mathbb{Z}$$ where p_i are prime.

~~Let $\tilde{X}^n = \bigvee_{m+k} S$~~. The approach is to wedge a bunch of spheres for each generator of A_i.

If $A_i \simeq \mathbb{Z}^m$ take a wedge of m S^i cells.

If $A_i \simeq \mathbb{Z}/p_j^{d_j}\mathbb{Z}$ take e^{i+1} with attaching map of degree $p_j^{d_j}$ to S^i.

2 (cont). To get $A_i \cong \mathbb{Z}^m \oplus \mathbb{Z}/p_1^{d_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k^{d_k}\mathbb{Z}$ take a wedge sum of spheres or e^{i+1} glued to a sphere, one for each generator. This gives an X_i s.t. $H_i(X_i) \cong A_i$, $H_k(X_i) = 0$ $k \neq i$.

To get an X s.t. $H_i(X) = A_i$ take $X = \vee X_i$

3. The LES gives

$$\begin{tikzcd} \cdots \arrow[r] & \cancel{H_n(CA)} \overset{0}{} \arrow[r] & H_n(X \cup CA) \arrow[d] \\ & & H_n(X \cup CA, CA) \arrow[d] \\ & & \cancel{H_{n-1}(CA)} \overset{0}{} \arrow[d] \\ & & \vdots \end{tikzcd}$$

so $H_n(X \cup CA) \cong H_n(X \cup CA, CA)$ as CA is contractible.

By excision $H_n(X \cup CA, CA) \cong H_n(X \cup (CA - pt), CA - pt)$

$\cong H_n(X, A)$

as $CA - pt$ retracts onto A.

4. WTS $f^*(\alpha) = 0$ for $\alpha \in H^2(T^2)$.

Let $[T^2] \in H^2(T^2)$, $[S^2] \in H^2(S^2)$ be the fundamental classes. If $f^*(\alpha) = 0$ then using the Kronecker pairing $H^2(X) \times H_2(X) \longrightarrow \mathbb{Z}$,

$0 = \langle f^*(\alpha), [S^2] \rangle = \langle \alpha, f_*[S^2] \rangle \Rightarrow f_*[S^2] = 0$

as $\langle , \rangle$ is nondegenerate ($\langle , \rangle$ is degenerate if $\exists$ $\beta \in H^i(X)$ s.t. $\langle \beta, \gamma \rangle = 0$ $\forall$ $\gamma \in H_i(X)$).

So wts $f^*(\alpha) = 0$.

~~Note~~ Let $\alpha_1, \alpha_2 \in H^1(T^2)$ be the generators. By ~~ex~~ 3.11 $\alpha_1 \cup \alpha_2 \in H^2(T^2)$ is the generator.

But $f^*(\alpha_1 \cup \alpha_2) = f^*(\alpha_1) \cup f^*(\alpha_2) = 0$ as ~~$f^*(\alpha_1)$~~

$f^*(\alpha_1) \in H^1(S^2) = 0$ $\Rightarrow$ $f^*(\alpha_1 \cup \alpha_2) = 0$ so done.

5. Done in class

6. $S^2 \times S^4$ has decomp $e^0 \times e^0$, $e^0 \times e^4$, $e^2 \times e^0$, $e^2 \times e^4$.

$$H_k(S^2 \times S^4) = \begin{cases} \mathbb{Z} & n = 0, 2, 4, 6 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow H^k(S^2 \times S^4) = \begin{cases} \mathbb{Z} & n = 0, 2, 4, 6 \\ 0 & \text{else} \end{cases}$$ by Poincare duality.

Let $\alpha_2 \in H^2(S^2 \times S^4)$, etc. Then $\alpha_0^\vee \in H_6(S^2 \times S^4)$

($\alpha_0^\vee$ is the Poincare dual of $\alpha_0 \in H^0(S^2 \times S^4)$) and $\alpha_0^\vee$ is $S^2 \times S^4$.

Likewise $\alpha_2^\vee$ is S^4, $\alpha_4^\vee$ is S^2, $\alpha_6^\vee$ is a point

$\alpha_2^\vee \cap \alpha_4^\vee$ is a point ($\alpha_2^\vee$ is $pt \times S^4$, $\alpha_4^\vee = S^2 \times pt$ so intersection is $pt \times pt$). $\Rightarrow$ $\alpha_2 \cup \alpha_4$ is a generator in H^6

However, ~~$\alpha_2 \cup \alpha_2$~~ $\alpha_2 \cup \alpha_2 = 0$ as $\alpha_2^\vee \cap \alpha_2^\vee = \emptyset$

(in general position — e.g. ~~if you~~ most ~~[illegible]~~ choices give $\emptyset$).

So $\alpha_2 \cup \alpha_2 = 0$. Likewise for α_4, α_6.

So $H^*(S^2 \times S^4) = \mathbb{Z}[\alpha_2, \alpha_4] / (\alpha_2^2, \alpha_4^2)$

But $H^*(\mathbb{CP}^n) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$ so $\mathbb{CP}^n \not\cong S^2 \times S^4$.

7. As $H^i(X, \mathbb{Q}) \times H_i(X, \mathbb{Q}) \longrightarrow \mathbb{Q}$ is

non degenerate $\Rightarrow$ $\dim H^i(X, \mathbb{Q}) = \dim H_i(X, \mathbb{Q})$

else $\exists$ $v \in H^i(X, \mathbb{Q}) - \{0\}$ (or $v \in H_i(X, \mathbb{Q}) - \{0\}$)

that maps to zero — i.e. the map is degenerate giving

a contradiction.

Hence $\dim H_i(X, \mathbb{Q}) = \dim H^i(X, \mathbb{Q}) = \dim H_{n-i}(X, \mathbb{Q})$

by Poincaré duality. Hence $\chi(X) = \sum (-1)^i \dim H_i(X) = 0$

as we can cancel out pairs.

8. γ non sep $\Rightarrow$ γ non zero.

If γ is non separating consider the following

is a closed simple loop (e.g. it doesn't intersect itself)

so we can get an annulus on it (or a very short cylinder)

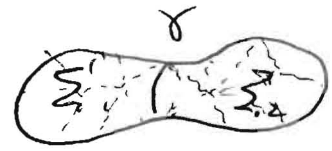
take two points, one on each boundary component. $\exists$ D, a loop connecting them

So $[\gamma].[D] = \pm 1$ (intersection multiplicity)

So $\gamma \neq 0$

γ sep $\Rightarrow$ γ zero

Triangulate the manifold so γ lies along the boundary of the triangles. It separates Σ into Σ_1, Σ_2



Since γ separates. Then γ is a boundary of Σ_1 so γ is zero

9. By the LES of a pair

$$\to H_i(\partial W) \to \overset{0}{\cancel{H_i(W)}} \to H_i(W,\partial W) \xrightarrow{\sim} H_{i-1}(\partial W) \to \overset{0}{\cancel{H_{i-1}(W)}} \to \cdots$$

$$\Rightarrow \quad H_i(W,\partial W) \simeq H_{i-1}(\partial W).$$

Lefshetz duality $\Rightarrow$ $H_i(W,\partial W) \simeq H^{4-i}(W)$

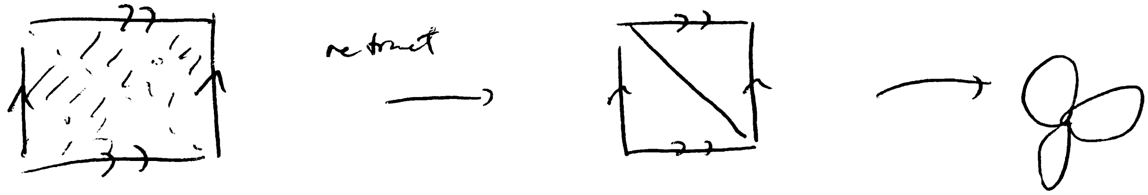
so $H_{i-1}(\partial W) \simeq H^{4-i}(W)$

$H_1(\partial W) = H^2(W) = 0$, $H_2(\partial W) = H^1(W) = 0$

$H_3(\partial W) = H^0(W) = \mathbb{Z}$

$H_0(\partial W) = \mathbb{Z}$ by path connectivity so ∂W is a homology sphere.

10. This space is $S^1 \vee S^1 \vee S^1$



So $H_*(X) = \begin{cases} \mathbb{Z}^3 & n=1 \\ \mathbb{Z} & n=0 \\ 0 & \text{else} \end{cases}$, $H^*(X) = \begin{cases} \mathbb{Z}^3 & n=1 \\ \mathbb{Z} & n=0 \\ 0 & \text{else} \end{cases}$

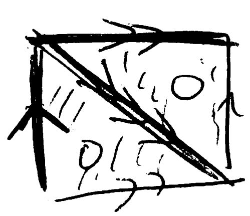
By Cor 3.3

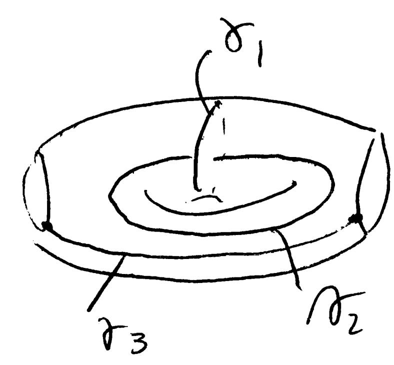
Use universal coeff Thm to get $H^*(X)$.

Use Lefschetz duality to get $H_*(X, \partial X)$, $H^*(X, \partial X)$

from $H^*(X)$, $H_*(X)$ respectively.

Generators (I believe) ~~are the loops~~ for X

are  the 3 bold lines.

Generators for $(X, \partial X)$ are 

From this the pairings are just the intersections