

Thanks to Rudy Perkins for supplying most of the solutions.

12. (a)

$$\begin{aligned}c_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-imx} dx \\&= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-imx} dx \\&= \frac{1}{2\pi} \left[\frac{-1}{im} e^{-imx} \right]_{-\delta}^{\delta} \\&= \frac{1}{m\pi} \left(\frac{e^{im\delta} - e^{-im\delta}}{2i} \right) \\&= \frac{\sin m\delta}{m\pi}\end{aligned}$$

(b)

$$\begin{aligned}1 &= f(0) \\&= \sum_{-\infty}^{\infty} \frac{\sin m\delta}{\pi m} e^{im0} \\&= \lim_{m \rightarrow 0} \frac{\sin m\delta}{\pi m} + \frac{2}{\pi} \sum_1^{\infty} \frac{\sin m\delta}{m} \\&= \frac{\delta}{\pi} + \frac{2}{\pi} \sum_1^{\infty} \frac{\sin m\delta}{m}\end{aligned}$$

which gives

$$\frac{\pi - \delta}{2} = \sum_1^{\infty} \frac{\sin m\delta}{m}.$$

(c) Using Parseval's theorem:

$$\begin{aligned}
 \frac{\delta}{\pi} &= \frac{1}{2\pi} \int_{-\delta}^{\delta} 1 \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \\
 &= \sum_{-\infty}^{\infty} \frac{\sin^2 m\delta}{\pi^2 m^2} \\
 &= \lim_{m \rightarrow 0} \frac{\sin^2 m\delta}{\pi^2 m^2} + \frac{2}{\pi^2} \sum_1^{\infty} \frac{\sin^2 m\delta}{m^2} \\
 &= \lim_{m \rightarrow 0} \frac{2\delta \sin m\delta \cos m\delta}{2\pi^2 m} + \frac{2}{\pi^2} \sum_1^{\infty} \frac{\sin^2 m\delta}{m^2} \\
 &= \lim_{m \rightarrow 0} \frac{2\delta^2 (\cos^2 m\delta - \sin^2 m\delta)}{2\pi^2} + \frac{2}{\pi^2} \sum_1^{\infty} \frac{\sin^2 m\delta}{m^2} \\
 &= \frac{\delta^2}{\pi^2} + \frac{2}{\pi^2} \sum_1^{\infty} \frac{\sin^2 m\delta}{m^2}
 \end{aligned}$$

In short we have

$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + \frac{2}{\pi^2} \sum_1^{\infty} \frac{\sin^2 m\delta}{m^2}.$$

Rearranging we get

$$\frac{\pi - \delta}{2} = \sum_1^{\infty} \frac{\sin^2 m\delta}{m^2 \delta}.$$

(d) By Thm 6.7c we have

$$\frac{\pi}{2} = \lim_{\delta \rightarrow 0} \sum_1^{\infty} \frac{\sin^2 m\delta}{(m\delta)^2} \cdot \delta = \int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx.$$

(e)

$$\sum_1^{\infty} \frac{\sin^2 n\pi/2}{m^2 \pi/2} = \frac{2}{\pi} \sum_1^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{4}$$

or

$$\sum_1^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

13. We calculate the Fourier coefficients to be

$$|c_n| = \left| \frac{2i\pi \cos m\pi}{m} - \frac{2i \sin m\pi}{m^2} \right| = \left| \frac{i(-1)^n}{m} \right| = \frac{1}{m}, \quad |c_0| = 0$$

Hence by Parseval's theorem:

$$\frac{\pi^2}{3} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2 = 2 \sum_1^{\infty} \frac{1}{n^2}$$

or

$$\frac{\pi^2}{6} = \sum_1^{\infty} \frac{1}{n^2}.$$

15. Recall from p. 189:

$$(e^{ix} - 1)D_n(x) = e^{i(n+1)x} - e^{-inx}$$

and note

$$\frac{1 - \cos(n+1)x}{1 - \cos x} = \frac{2 - e^{i(n+1)x} - e^{-i(n+1)x}}{2 - e^{ix} - e^{-ix}}.$$

Hence the problem reduces to showing

$$(2 - e^{i(N+1)x} - e^{-i(N+1)x}) = (2 - e^{ix} - e^{-ix}) \sum_0^N D_n.$$

Using the information above, we get:

$$\begin{aligned} (2 - e^{ix} - e^{-ix}) \sum_0^N D_n &= \sum_0^N ((1 - e^{ix}) + (1 - e^{-ix})) D_n \\ &= \sum_0^N (1 - e^{ix}) D_n + \sum_0^N (1 - e^{-ix}) D_n \\ &= \sum_0^N (e^{-inx} - e^{i(n+1)x}) + \sum_0^N (e^{inx} - e^{-i(n+1)x}) \\ &= \sum_0^N (e^{inx} - e^{i(n+1)x}) + \sum_0^N (e^{-inx} - e^{-i(n+1)x}) \\ &= \sum_0^N (e^{inx} - e^{i(n+1)x}) + \sum_0^N (e^{-inx} - e^{-i(n+1)x}) \\ &= (1 - e^{i(N+1)x}) + (1 - e^{-i(N+1)x}) \\ &= (2 - e^{i(N+1)x} - e^{-i(N+1)x}) \end{aligned}$$

(a) If $x \neq 2n\pi$ then

$$1 - \cos(N+1)x \geq 0, \quad 1 - \cos x \geq 0 \Rightarrow K_n \geq 0.$$

If $x = 2n\pi$ then we show that $\lim K_n > 0$:

$$\begin{aligned}
\lim_{x \rightarrow 2n\pi} K_n &= \frac{1}{N+1} \lim_{x \rightarrow 2n\pi} \frac{1 - \cos(N+1)x}{1 - \cos x} \\
&= \frac{1}{N+1} \lim_{x \rightarrow 2n\pi} \frac{-(N+1) \sin(N+1)x}{-\sin x} \\
&= \lim_{x \rightarrow 2n\pi} \frac{\sin(N+1)x}{\sin x} \\
&= \lim_{x \rightarrow 2n\pi} \frac{(N+1) \cos(N+1)x}{\cos x} \\
&= N+1 \\
&> 0
\end{aligned}$$

(b) Recall

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n = 1$$

so we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{N+1} \sum_0^N D_n = \frac{1}{N+1} \cdot (N+1) = 1.$$

(c) Note for $0 < \delta \leq |x| \leq \pi$ we have $1 - \cos x \geq 1 - \cos \delta$. Hence

$$K_N = \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x} \leq \frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta} \leq \frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta}.$$

Recall

$$s_n(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \pi f(x-t) D_n dt$$

so we obtain

$$\begin{aligned}
\sigma_N(f; x) &= \frac{s_0 + \cdots + s_N}{N+1} \\
&= \sum_0^N \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n dt}{N+1} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \sum_0^N \frac{D_n}{N+1} dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_n dt
\end{aligned}$$

To prove Fejér's theorem, we need to show $\sup |\sigma_N - f| \rightarrow 0$ as $N \rightarrow \infty$.

$$\begin{aligned} |\sigma_N(f; x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_n dt - f(x) \right| \\ &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x)) K_n dt \right| \text{ (from (b))} \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)| K_n dt \end{aligned}$$

Since f is continuous on a compact set, we can choose δ s.t.

$$|x - y| < 2\delta \Rightarrow |f(x) - f(y)| \leq \epsilon/3.$$

Let $M = \sup |f|$:

$$\begin{aligned} |\sigma_N(f; x) - f(x)| &\leq \frac{1}{2\pi} \int_{-\pi}^{-\delta} 2MK_n dt + \frac{1}{2\pi} \int_{-\delta}^{\delta} (\epsilon/3) K_n dt + \frac{1}{2\pi} \int_{\delta}^{\pi} 2MK_n dt \\ &\leq \frac{2M}{\pi} \left(\frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta} \right) + \epsilon/3 \\ &< \epsilon \end{aligned}$$

if we take $N \gg 0$.

16. Let $\epsilon > 0$. We want an N s.t.

$$|\sigma_n(f; x) - \frac{1}{2}[f(x+) - f(x-)]| < \epsilon, \quad n \geq N.$$

Let $M = \sup |f|$. Recalling the results from the previous exercise,

$$\begin{aligned}
\left| \sigma_n(f; x) - \frac{1}{2}[f(x+) - f(x-)] \right| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_n dt - \frac{1}{2}[f(x+) + f(x-)] \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(x-t) - \frac{1}{2}[f(x+) + f(x-)] \right) K_n dt \right| \\
&\leq \left| \frac{1}{2\pi} \int_{-\pi}^{-\delta} \left(f(x-t) - \frac{1}{2}[f(x+) + f(x-)] \right) K_n dt \right| \\
&\quad + \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} \left(f(x-t) - \frac{1}{2}[f(x+) + f(x-)] \right) K_n dt \right| \\
&\quad + \left| \frac{1}{2\pi} \int_{\delta}^{\pi} \left(f(x-t) - \frac{1}{2}[f(x+) + f(x-)] \right) K_n dt \right| \\
&\leq \left| \frac{1}{2\pi} \int_{-\pi}^{-\delta} 2M K_n dt \right| \\
&\quad + \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} \left(f(x-t) - \frac{1}{2}[f(x+) + f(x-)] \right) K_n dt \right| \\
&\quad + \left| \frac{1}{2\pi} \int_{\delta}^{\pi} 2M K_n dt \right|
\end{aligned}$$

Using the logic of Ex 6.12, we can choose a $g \in C^0$ s.t.

$$\int_{-\pi}^{\pi} |f - g| K_n < (2\pi\epsilon)/6.$$

From the construction of g we can choose $\delta > 0$ s.t.

$$|t| < \delta \Rightarrow \left| g(x-t) - \frac{1}{2}[f(x+) + f(x-)] \right| < (2\pi\epsilon)/6.$$

Hence the middle integral gives

$$\begin{aligned}
\left| \frac{1}{2\pi} \int_{-\delta}^{\delta} \left(f(x-t) - \frac{1}{2}[f(x+) + f(x-)] \right) K_n dt \right| &\leq \frac{1}{2\pi} \int_{-\delta}^{\delta} |f(x-t) - g(x-t)| K_n dt \\
&\quad + \frac{1}{2\pi} \int_{-\delta}^{\delta} \left| g(x-t) - \frac{1}{2}[f(x+) + f(x-)] \right| K_n dt \\
&< \epsilon/6 + \epsilon/6 \\
&= \epsilon/3
\end{aligned}$$

From 15(c), we can choose $N \gg 0$ s.t.

$$\left| \frac{1}{2\pi} \int_{-\pi}^{-\delta} 2M K_n dt \right| < \epsilon/3, \quad \left| \frac{1}{2\pi} \int_{\delta}^{\pi} 2M K_n dt \right| < \epsilon/3.$$

Hence we get

$$\left| \sigma_n(f; x) - \frac{1}{2}[f(x+) - f(x-)] \right| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

17. (a) Recall from Ex 6.17 that if f is bounded and monotonically increasing (if decreasing, replace by $-f$), g is continuous and has an antiderivative G then we have

$$\int_a^b f(x)g(x)dx = G(b)f(b) - G(a)f(a) - \int_a^b Gdf.$$

Take $g = e^{-inx}$ and $G = ie^{-inx}/n$ and consider the interval $[-\pi, \pi - \delta]$. Then we get

$$\begin{aligned} c_n &= \lim_{\delta \rightarrow 0} \int_{-\pi}^{\pi-\delta} f(x)g(x)dx \\ &= \lim_{\delta \rightarrow 0} f(\pi - \delta)G(\pi - \delta) - f(-\pi)G(-\pi) - \int_{-\pi}^{\pi-\delta} Gdf \\ &= f(\pi)G(\pi) - f(-\pi)G(-\pi) - \int_{-\pi}^{\pi} Gdf \end{aligned}$$

Hence

$$\begin{aligned} |nc_n| &\leq n \left| \frac{f(\pi)e^{-in\pi}}{n} \right| + n \left| \frac{f(-\pi)e^{in\pi}}{n} \right| + n \int_{-\pi}^{\pi} \left| \frac{e^{-inx}}{n} \right| df \\ &\leq |f(\pi)| + |f(-\pi)| + \int_{-\pi}^{\pi} 1df \\ &= |f(\pi)| + |f(-\pi)| + (f(\pi) - f(-\pi)) \end{aligned}$$

In short, $|nc_n| < M$.

- (b) Ex 3.14e says if $|nc_n| < M$ then $\lim \sigma_n = \lim s_n$. But Ex 8.16 says $\lim \sigma_n = \frac{1}{2}[f(x+) + f(x-)]$ hence

$$\lim s_n = \frac{1}{2}[f(x+) + f(x-)].$$

- (c) Consider the function

$$g = \begin{cases} f(\alpha) & \text{if } -\pi \leq x < \alpha \\ f(x) & \text{if } \alpha \leq x \leq \beta \\ f(\beta) & \text{if } \beta < x \leq \pi \end{cases}$$

This is monotonic and bounded hence the previous parts apply. By the localization theorem (p. 190) we have

$$\lim[s_N(f; x) - s_N(g; x)] = 0, \quad x \in (\alpha, \beta).$$

Since the one sided limits of f and g agree in (α, β) we have

$$\lim s_N(f; x) = \lim s_N(g; x) = \frac{1}{2}[g(x+) + g(x-)] = \frac{1}{2}[f(x+) + f(x-)].$$