DERIVED FUNCTORS

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Unless otherwise stated, all categories are abelian and all functors are covariant. $\mathfrak{Ab}$ is the category of abelian groups, $\mathfrak{Mod}(X)$ is sheaves of modules over X and $\mathfrak{Ab}(X)$ is sheaves of abelian groups over X.

1 DERIVED FUNCTORS.

An *injective resolution* of $A \in \mathfrak{A}$ is an exact sequence $0 \to A \to I^{\cdot}$ where $I^j \in \mathfrak{A}$ is injective. Given a left-exact functor $F : \mathfrak{A} \to \mathfrak{B}$, define the *right derived functors* $R^iF : \mathfrak{A} \to \mathfrak{B}$ by taking an arbitrary injective resolution of an object and setting $R^iF(A) = h^i(F(I^{\cdot}))$.

2 INJECTIVE RESOLUTIONS.

We start with a definition: $\mathfrak{A}$ has *enough injectives* if every object can be embedded into an injective object—e.g. for any $A \in \mathfrak{A}$ there is an injective object $I \in \mathfrak{A}$ and a monomorphism $u : A \hookrightarrow I$.

Theorem 1 *If $\mathfrak{A}$ has enough injectives then every object has an injective resolution.*

Instead of asking when does an object have an injective resolution, we generalize to asking when does a category have enough injectives? If X is a ringed space then $\mathfrak{Mod}(X)$ and $\mathfrak{Ab}(X)$ have enough injectives. If X is a noetherian scheme then $\mathfrak{Qco}(X)$ has enough injectives (Hartshorne, Ex. III.3.6a). $\mathfrak{Coh}(X)$ doesn't have enough injectives, though it has enough locally frees (ibid., Ex. III.6.4).

As an aside, we commonly believe if a space has enough injectives then it probably has enough projectives. This certainly holds in R-**mod** but not in general. For instance $\mathfrak{Mod}(\mathbb{P}^1_k)$ and $\mathfrak{Qco}(\mathbb{P}^1_k)$ don't have enough projectives even though $\mathbb{P}^1_k$ is certainly very nice (ibid., Ex. III.6.2).

It might seem that choosing different resolutions will cause problems but this isn't the case:

Theorem 2 *Two resolutions of the same object will be chain homotopic. In particular, their cohomology will be the same.*

3 COVARIANT δ-FUNCTORS.

Definition Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories. A *covariant δ-functor* is a collection of functors $T = (T^i) : \mathfrak{A} \to \mathfrak{B}$ along with $\delta^i : T^i(A'') \to T^{i+1}(A')$ for each short exact sequence $0 \to A' \to A \to A'' \to 0$ such that

1. there exists a long exact sequence with δ^i acting as connecting homomorphisms, and
2. maps between short exact sequences gives a commutative ladder.

Example $T^i = H^i(X, \cdot) : \mathfrak{Mod}(X) \to \mathfrak{Ab}$ is a covariant δ-functor.

Non-example Let $\mathfrak{U} = \{X\}$ be the trivial cover and consider the the Čech cohomology functor. $T^0 = \check{H}^0(\mathfrak{U}, \cdot) : \mathfrak{Mod}(X) \to \mathfrak{Ab}$ is simply taking global sections and $T^i = 0$ by construction. Since taking global sections isn't right exact, Čech cohomology isn't a δ-functor.

4 UNIVERSAL δ-FUNCTORS.

Definition $T : \mathfrak{A} \to \mathfrak{B}$ is a universal δ-functor if given

1. another δ-functor $T' : \mathfrak{A} \to \mathfrak{B}$
2. a natural transformation $f^0 : T^0 \to T'^0$

then there exists unique $f^i : T^i \to T'^i$ which commute with the connecting homomorphisms for each short exact sequence.

If two δ-functors are universal with $T^0 \cong T'^0$ then we can conclude $T^i \cong T'^i$. For instance, if we want to show two methods of computing cohomology give the same result then we need only show they agree at the zeroth level and are universal.

Example Let $V \subset X$ be open and $Y \subset V$ closed in X. Suppose we want to prove excision:

$$H^i_Y(X, \mathscr{F}) \cong H^i_Y(V, \mathscr{F}|_V).$$

By showing (somehow) that $H^0_Y(X, \mathscr{F}) \cong H^0_Y(V, \mathscr{F}|_V)$ and that both are universal δ-functors, we obtain the desired isomorphisms.

From the definition alone, it would be hard to determine if a δ-functor is universal. However there is a way to get a handle on it; first we need a definition.

Definition A functor $F : \mathfrak{A} \to \mathfrak{B}$ is *effaceable* if for all $C \in \mathfrak{A}$ there exists an $M \in \mathfrak{A}$ and a monomorphism $u : C \to M$ s.t. $F(u) = 0$.

Effaceable means to rub out or erase—e.g. getting rid of cohomology for $i > 0$.

Proposition 3 *Let T be a δ-functor. If T^i is effaceable for $i > 0$ then T^i is universal.*

Example H^i is effaceable: for any $\mathscr{F} \in \mathfrak{Mod}(X)$ we can find an injective object $\mathscr{I}$ with $\mathscr{F} \hookrightarrow \mathscr{I}$ and $H^i(X, \mathscr{I}) = 0$ for $i > 0$.

In fact δ-functors and derived functors are closely related.

Corollary 4 *Suppose $\mathfrak{A}$ has enough injectives and let $T : \mathfrak{A} \to \mathfrak{B}$ be a δ-functor. Then T is universal iff T^0 is left exact and $T^i \cong R^i T^0$.*

5 GROTHENDIECK CATEGORIES.

From Grothendieck's Tohoku paper, we can give a criterion for when a category has enough injectives. However there is a lot of baggage that comes along the way. $U \in \mathfrak{A}$ is a *generator* if for any map $A \xrightarrow{g} B$ in $\mathfrak{A}$ there is a map $U \xrightarrow{f} A$ with $fg \neq 0$. For example, if $\mathfrak{A} = R$-**mod** then R is a generator. A harder example is that $\mathfrak{Ab}(X)$ has a generator, which I won't prove.

A category is *cocomplete* if colimits exist; the best way to think about this is to ask if the category is closed under direct sums. For example, finite dimensional vector spaces are not cocomplete. R-**mod** is cocomplete.

We say a category is *AB5* if it is cocomplete and for a directed family of subobjects $\{U_i\}_I$ of an object A and for any subobject V we have

$$\big(\bigcup_I U_i\big) \cap V = \bigcup_I \big(U_i \cap V\big).$$

An AB5 category with a generator is called a *Grothendieck category*.

Theorem 5 (Tohoku 1.10.1) *If $\mathfrak{A}$ is a Grothendieck category then it has enough injectives.*