A NOTE ON THE EXISTENCE OF EXOTIC 7-SPHERES

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This note is based on a lecture given by Nitu Kitchloo at UCSD in Winter, 2005. The purpose is to show the existence of a smooth manifold that is homeomorphic, but not diffeomorphic, to S^7. The method loosely follows that of Milnor's 1956 argument.

1. Preliminaries

We'll need a couple facts (not proven here) before we get started. The first tells us about the pontrjagin classes of complex projective spaces.

Fact 1.1. $p_i(T\mathbb{C}P^n) = \binom{n+1}{i}e^{2i}$*, where* e *is the euler class of the canonical line bundle.*

We also need some information about the oriented cobordism group of 8-manifolds:

Fact 1.2. $\Omega_8^{or} = \mathbb{Z}\langle\mathbb{C}P^4\rangle \ \oplus \ \mathbb{Z}\langle\mathbb{C}P^2 \times \mathbb{C}P^2\rangle$

Here's the game plan: in Section 2 we'll examine some properties of smooth 8-manifolds. Then in Section 3 we shall consider the disk bundle of a certain vector bundle (to be constructed in Section 4), the total space of which has boundary homeomorphic to the 7-sphere. Assuming it's in fact diffeomorphic, we can construct a smooth 8-manifold from the disk bundle that fails to support the properties from Section 2, giving us a contradiction.

2. Characteristic Numbers and Signature

Here we'll prove some results about $4n$-manifolds in the context of cobordism. Consider the polynomial ring $\mathbb{Z}[p_1, p_2, ..., p_n]$, where the formal degree of p_i is $4i$. Let $q \in \mathbb{Z}[p_1, p_2, ..., p_n]$ be a polynomial of homogeneous degree $4n$ for some fixed n. (For example, $q = p_2 - p_1^2$ where $n = 2$.)

Definition 2.1. *Given such a polynomial* q *and a smooth oriented* $4n$*-manifold* M^{4n}*, define the* characteristic number $q(M) \in \mathbb{Z}$ *as*

$$q(M) = \langle q(p_1(TM), p_2(TM), ..., p_n(TM)); [M]\rangle$$

where $p_i(TM)$ *is the* i^{th} *pontrjagin class of the tangent bundle of* M*,* $[M] \in H_{4n}(M;\mathbb{Z})$ *is the fundamental class, and* $\langle \ ; \ \rangle$ *is the Kronecker pairing.*

First we'll show that characteristic numbers are invariant under cobordism.

Proposition 2.2. *If* M^{4n} *is cobordant to* N^{4n}*, then* $q(M) = q(N)$*. In fact,* $q : \Omega_{4n}^{or} \to \mathbb{Z}$ *is a homomorphism of abelian groups.*

Date: May 19, 2005.

Proof. It is an easy check that the map q respects additive structure, since addition in the cobordism group is just disjoint union.

Now say W is a $(4n+1)$-manifold with $\partial W = M - N$, the disjoint union of M and N (the latter being oriented negatively). Note that $TW \mid_{\partial W} = T\partial W \oplus \ \ell$, where ℓ is the trivial bundle of outward-pointing normal vectors (see Figure 1). Let $i : \partial W \hookrightarrow W$ be the inclusion. Then $i^* p_i(TW) = p_i(T\partial W)$ from naturality of pontrjagin classes, and the fact of their stability under Whitney sum with a trivial bundle. So we have that

$$i^* q(p_1(TW), ..., p_n(TW)) = q(p_1(T\partial W), ..., p_1(T\partial W))$$

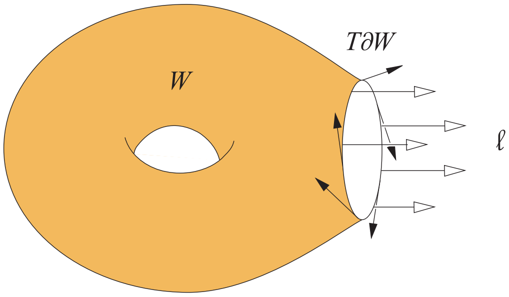

FIGURE 1

From our earlier observation, $q(\partial W) = q(M) - q(N)$. Thus

$$\begin{aligned} q(M) - q(N) &= \langle q(p_1(T\partial W), ..., p_n(T\partial W)); [\partial W]\rangle \\ &= \langle i^* q(p_1(TW), ..., p_n(TW)); [\partial W]\rangle \\ &= \langle q(p_1(TW), ..., p_n(TW)); i_*[\partial W]\rangle \\ &= 0 \end{aligned}$$

using the fact that $i_*[\partial W] = 0$ from the homology long exact sequence for the pair $(W, \partial W)$, in which the fundamental class $[W, \partial W]$ maps to $[\partial W]$. This gives the result.

□

Example. Let $n = 2$ and $q = 7p_2 - p_1^2$, so we have a map $q : \Omega_8^{or} \to \mathbb{Z}$. Let's compute q for the generators, which we know from **Fact 1.2**. Recall that $H^*(\mathbb{C}P^4; \mathbb{Z}) = \mathbb{Z}[e]/\langle e^5\rangle$, and e^4 generates the top cohomology. **Fact 1.1** tells us that $p_1(T\mathbb{C}P^4) = 5e^2$ and $p_2(T\mathbb{C}P^4) = 10e^4$. Thus

$$q(\mathbb{C}P^4) = \ \langle (70 - 25)e^4; [\mathbb{C}P^4]\rangle \ = 45$$

We also have from **Fact 1.1** that $p_1(T\mathbb{C}P^2) = 3e^2$. View the external Whitney sum bundle over $\mathbb{C}P^2 \times \mathbb{C}P^2$ as $(T\mathbb{C}P_1^2 \oplus 0) \oplus (0 \oplus T\mathbb{C}P_2^2)$, an internal Whitney sum. Since $H^*(\mathbb{C}P^2 \times \mathbb{C}P^2; \mathbb{Z})$ has no 2-torsion, the Whitney sum formula gives

$$p_1(T(\mathbb{C}P^2 \times \mathbb{C}P^2)) = p_1(T\mathbb{C}P_1^2 \oplus 0) + p_1(0 \oplus T\mathbb{C}P_2^2) = 3e_1^2 + 3e_2^2$$

and

$$p_2(T(\mathbb{C}P^2 \times \mathbb{C}P^2)) = p_1(T\mathbb{C}P_1^2 \oplus 0) \smile p_1(0 \oplus T\mathbb{C}P_2^2) = 9e_1^2e_2^2$$

Since $e_1^2e_2^2$ generates the top cohomology of $\mathbb{C}P^2 \times \mathbb{C}P^2$ by the Künneth formula, we get

$$q(\mathbb{C}P^2 \times \mathbb{C}P^2) = \langle 63e_1^2e_2^2 - 18e_1^2e_2^2; [\mathbb{C}P^2 \times \mathbb{C}P^2]\rangle = 45$$

In fact, though it's not strictly relevant here, we discover that $\frac{1}{45}(7p_2 - p_1^2) : \Omega_8^{or} \to \mathbb{Z}$ is a surjective group homomorphism.

Now we'll define the signature of a manifold and see how it relates to the map we've just considered. Let M^{4n} be an oriented manifold. There is a symmetric nondegenerate bilinear pairing $Q : H^{2n}(M;\mathbb{Q}) \times H^{2n}(M;\mathbb{Q}) \to \mathbb{Q}$, sending $(\alpha,\beta) \mapsto \langle \alpha \smile \beta, [M]\rangle$.

Definition 2.3. *The* signature of $\mathrm{M}^{4\mathrm{n}}$, *$\sigma(M) \in \mathbb{Z}$, is defined to be $\sigma(M) = p^+ - p^-$, where p^+ (resp. p^-) is the number of positive (resp. negative) eigenvalues of Q.*

As one might wish, $\sigma(M)$ is also invariant under cobordism.

Proposition 2.4. *If M^{4n} is cobordant to N^{4n}, then $\sigma(M) = \sigma(N)$*

To prove this, we'll first need a general result from linear algebra.

Lemma 2.5. *Let $Q : V \times V \to \mathbb{F}$ be a symmetric nondegenerate bilinear form on a finite-dimensional real vector space V, and let $W = \{x \in V : Q(x,x) = 0\}$. If $\dim W = \frac{1}{2}\dim V$, then the matrix of Q has an equal number of positive and negative eigenvalues.*

Proof. We adopt this proof from [Co]. Since Q is nondegenerate, we have a decomposition $V = V_+ \oplus V_-$, where Q is positive definite on V_+ and negative definite on V_-. (For more details, see Sec. 9.5 of [Ro]). Let $s_+ = \dim V_+$ and $s_- = \dim V_-$, which are the number of positive (resp. negative) eigenvalues of Q.

Note that W is a subspace of V, and it's easily checked that $W \cap V_\pm = \emptyset$. Let $W_1 = W \oplus V_+$, and let $W_2 = W \oplus V_-$. Then, if $\dim V = n$,

$$\dim W_1 = \frac{n}{2} + s_+ \leq n \;\Rightarrow\; s_+ \leq \frac{n}{2}$$
$$\dim W_2 = \frac{n}{2} + s_- \leq n \;\Rightarrow\; s_- \leq \frac{n}{2}$$

But since $s_+ + s_- = n$, we must have that $s_+ = s_- = \frac{n}{2}$.

□

Proof. (of Proposition 2.4). This proof is adopted from [Hi]. Note that $\sigma(M \amalg N) = \sigma(N) + \sigma(M)$, since $H^{2n}(M \amalg N;\mathbb{Q}) = H^{2n}(M;\mathbb{Q}) \oplus H^{2n}(N;\mathbb{Q})$ and the matrix of Q will have two diagonal blocks, one for the restriction of Q to each manifold. It's also clear that $\sigma(-M) = -\sigma(M)$, since the fundamental homology class satisfies $-[M] = [-M]$.

Now it suffices to show that if $M - N = \partial W$ for some oriented $(4n+1)$-manifold W, then $\sigma(\partial W) = 0$. Consider the following commuting diagram:

$$\begin{tikzcd}
H^{2n}(W;\mathbb{Q}) \arrow[r, "i^*"] \arrow[d, "\mathtt{PD}"] & H^{2n}(\partial W;\mathbb{Q}) \arrow[r] \arrow[d, "\mathtt{PD}"] & H^{2n}(W,\partial W;\mathbb{Q}) \arrow[d, "\mathtt{PD}"] \\
H_{2n+1}(W,\partial W;\mathbb{Q}) \arrow[r] & H_{2n}(\partial W;\mathbb{Q}) \arrow[r, "i_*"] & H_{2n}(W;\mathbb{Q})
\end{tikzcd}$$

where the rows are (co)homology long-exact sequences for the pair $(W, \partial W)$, and the vertical arrows are isomorphisms via Poincaré Duality. Let $A^{2n} := im(i^*) \subset H^{2n}(\partial W; \mathbb{Q})$, and let $K_{2n} := ker(i_*) \subset H_{2n}(\partial W; \mathbb{Q})$. The exactness of the above diagram tells us that $A^{2n} \cong K_{2n}$.

But since we're working over a field, the universal coefficients theorem gives another commuting diagram

$$\begin{tikzcd} H^{2n}(W;\mathbb{Q}) \arrow[r, "i^*"] \arrow[d, "\cong"] & H^{2n}(\partial W;\mathbb{Q}) \arrow[d, "\cong"] \\ H_{2n}(W;\mathbb{Q}) & H_{2n}(\partial W;\mathbb{Q}) \arrow[l, "i_*"] \end{tikzcd}$$

Since the composition of the three maps on the right must be an isomorphism, we see that $A^{2n} \cong H_{2n}(\partial W; \mathbb{Q})/K_{2n}$. But now $\dim K_{2n} = \dim A^{2n} = b_{2n}(\partial W) - \dim K_{2n}$, where $b_{2n}(\partial W)$ is the $2n^{th}$ betti number of ∂W. Hence $\dim A^{2n} = \frac{1}{2} b_{2n}(\partial W)$,

Let $x \in A^{2n}$, so $x = i^* y$ for some $y \in H^{2n}(W; \mathbb{Q})$. Then

$$x^2[\partial W] = (i^* y)[\partial W] = y^2(i_*[\partial W]) = 0$$

since, as in the proof of **Prop. 2.2**, $[\partial W]$ is in the image of the previous map in the homology long exact sequence. Thus the subspace $\{x \in H^{2n}(\partial W; \mathbb{Q}) : Q(x,x) = \langle x^2; [\partial W] \rangle\}$ contains another subspace A^{2n} of dimension $\frac{1}{2} b_{2n}(\partial W)$. Applying **Lemma 2.5** gives the result.

□

So we're ready to make a key statement about our example from earlier in the section.

Proposition 2.6. (Hirzebruch) *Let M^8 be a compact, oriented 8-manifold. Then*

$$\sigma(M^8) = \frac{1}{45}(7p_2 - p_1^2)[M^8]$$

Proof. Having proven **Prop. 2.4**, it's sufficient to show the statement holds for $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$, the additive generators of Ω_8^{gr}.

In the first case, we have that $H^4(\mathbb{C}P^4; \mathbb{Q}) = \mathbb{Q}\langle e^2 \rangle$. So, since $\langle e^4; [\mathbb{C}P^4] \rangle = 1$, the matrix for Q here is just $(\ 1\ )$. Thus $\sigma(\mathbb{C}P^4) = 1 = \frac{1}{45}(7p_2 - p_1^2)[\mathbb{C}P^4]$, the second equality coming from our example.

For the second generator of Ω_8^{gr}, the Künneth formula says that

$$H^4(\mathbb{C}P^2 \times \mathbb{C}P^2; \mathbb{Q}) = \mathbb{Q}\langle e_1^2 \rangle \oplus \mathbb{Q}\langle e_2^2 \rangle \oplus \mathbb{Q}\langle e_1 e_2 \rangle.$$

Using the fact that $e_i^3 = 0$, a quick computation shows that the matrix for Q here is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So again we discover that $\sigma(\mathbb{C}P^2 \times \mathbb{C}P^2) = 1 = \frac{1}{45}(7p_2 - p_1^2)[\mathbb{C}P^2 \times \mathbb{C}P^2]$.

□

3. An Exotic 7-Sphere

This is where the fun really starts. Our candidate 7-spheres will turn out to be the total spaces of certain fiber bundles.

Consider a smooth oriented 4-dimensional real vector bundle $E \to S^4$. Since we can always put a Euclidean metric on such a bundle, it makes sense to talk about the corresponding sphere bundle $S(E) \to S^4$, where $S(E) \simeq (E - s_0)$ (a homotopy equivalence), s_0 being the image of the zero-section. This is a fiber bundle with fiber $S^3 = S(\mathbb{R}^4)$.

A familiar example, of course, is the Hopf fibration. It turns out that a simple condition on $S(E)$ makes this space, in one sense, just as familiar.

Proposition 3.1. *If $e \in H^4(S^4;\mathbb{Z})$, the euler class of $E \to S^4$, is a generator, then $S(E) \cong S^7$ (a homeomorphism).*

Proof. Recall the Gysin long-exact sequence for an oriented n-dimensional vector bundle $E \xrightarrow{\pi} B$, where $\tilde{\pi}$ is the restriction $(E - s_0) \to B$:

$$\begin{tikzcd}\cdots \arrow[r] & H^j(B;\mathbb{Z}) \arrow[r, "\smile e"] & H^{j+n}(B;\mathbb{Z}) \arrow[r, "\tilde{\pi}^*"] & H^{j+n}((E-s_0);\mathbb{Z}) \arrow[r] & \cdots\end{tikzcd}$$

In our case we get:

$$\begin{tikzcd}\cdots \arrow[r] & H^j(S^4;\mathbb{Z}) \arrow[r, "\smile e"] & H^{j+4}(S^4;\mathbb{Z}) \arrow[r, "\tilde{\pi}^*"] & H^{j+4}(S(E);\mathbb{Z}) \arrow[r] & \cdots\end{tikzcd}$$

Since e is a generator of $H^4(S^4;\mathbb{Z})$, the map $\smile e$ is an isomorphism for $j = 0$. This fact, along with the plethora of 0's in the sequence, yields the following:

$$H^i(S(E);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i\text{=0,7;} \\ 0 & \text{otherwise} \end{cases}$$

Poincaré Duality tells us this is true for homology as well. If $[f]$ generates $\pi_7(S(E))$, the Hurewicz isomorphism sends $[f]$ to $f_*[S^7]$, which must generate $H_7(S(E);\mathbb{Z})$. Thus $f : S^7 \to S(E)$ induces isomorphisms on homology. A theorem of Whitehead states then that f is actually a homotopy equivalence: $S(E) \simeq S^7$.

Now we can pull out the big guns. The Poincaré Theorem (formerly Conjecture) for $n \geq 5$, proven by Smale, says that a manifold homotopy equivalent to S^7 is actually homeomorphic to S^7. (Milnor's pre-Smale argument uses Morse theory.) □

Claim 3.2. *Given any $k \equiv 2 \pmod 4$, there exists a smooth oriented real 4-dimensional vector bundle $E \to S^4$ such that* (i) *the euler class e generates $H^4(S^4;\mathbb{Z})$, and* (ii) *the first pontrjagin class of the bundle satisfies $p_1 = ke \in H^4(S^4;\mathbb{Z})$.*

The proof is a little tedious, so to avoid killing our buzz we'll save it for the last section.

Note that in any such vector bundle, **Prop. 3.1** tells us that $S(E) \cong S^7$. Let's consider the disk bundle $D(E)$, a smooth 8-manifold with $\partial D(E) = S(E)$. *Now assume that $S(E)$ is diffeomorphic to S^7.* Then we can "complete" $D(E)$ to a closed oriented smooth 8-manifold M^8 by gluing in the 8-disk along $S(E) \cong S^7$ (see Figure 2).

The cohomology long-exact sequence for the pair $(M^8, D(E))$ tells us that

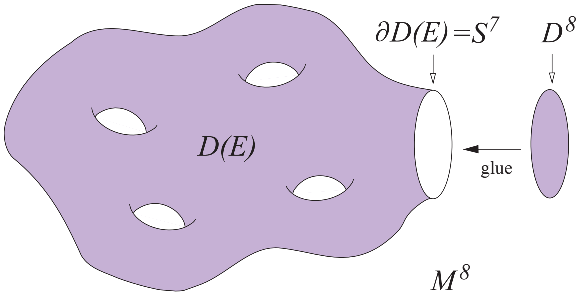

FIGURE 2

$$H^i(M^8;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i\text{=0,4,8;} \\ 0 & \text{otherwise} \end{cases}$$

Claim 3.3. *$p_1(TM^8) = ku$, where u generates $H^4(M^8;\mathbb{Z})$.*

Proof. Consider the following inclusions

$$S^4 \overset{s_0}{\hookrightarrow} D(E) \overset{i}{\hookrightarrow} M^8$$

where $(s_0)^*$ and i^* are isomorphisms on cohomology in dimensions one through four (via homotopy equivalence and the long exact sequence, respectively). Also, note that the restriction of the tangent bundle of M^8 to S^4 can be decomposed in the following way:

$$TM^8\mid_{S^4} \;=\; TD(E)\mid_{S^4} \;=\; TS^4 \oplus E$$

We thus have the following bundle morphism:

$$\begin{tikzcd} TS^4 \oplus E \arrow[r] \arrow[d] & TM^8 \arrow[d] \\ S^4 \arrow[r, "i \circ s_0"] & M^8 \end{tikzcd}$$

By naturality of the pontrjagin class, and the fact that $(i \circ s_0)^* : H^4(M^8;\mathbb{Z}) \to H^4(S^4;\mathbb{Z})$ is an isomorphism, we know that $p_1(TM^8)$ maps to $p_1(TS^4 \oplus E)$. The Whitney sum formula (there being no 2-torsion) says:

$$\begin{aligned} p_1(TS^4 \oplus E) &= p_0(TS^4) \smile p_1(E) + p_1(TS^4) \smile p_0(E) \\ &= p_1(E) \\ &= ke \end{aligned}$$

where the second equality is given by the fact that $p_1(TS^4) = 0$. (This statement is true because pontrjagin classes are invariant under Whitney sum with trivial bundles, and $TS^4 \oplus \eta$, where η is the 1-dimensional (trivial) normal bundle, is just the trivial 5-dimensional bundle over S^4. Since p_1 of this bundle is zero, so is $p_1(TS^4)$.)

Thus $p_1(TM^8)$ maps to ke. But since e generates $H^4(S^4;\mathbb{Z})$ and $(i \circ s_0)^*$ is an isomorphism, we must have that $p_1(TM^8) = ku$ where u is a generator of $H^4(M^8;\mathbb{Z})$.

□

Note that u^2 generates the top cohomology of M^8. We can see this using Poincaré duality: u is dual to $u \frown [M^8]$, which generates $H_4(M^8;\mathbb{Z})$, and thus coincides with $\pm$ the Hom dual of u. Thus $\pm 1 = \langle u, u \frown [M^8]\rangle = \langle u \smile u, [M^8]\rangle$. Let's choose the orientation on M that gives us the positive value.

Since $H^4(M^8;\mathbb{Q}) = \mathbb{Q}\langle u^2\rangle$, the matrix for Q (as defined before) is just $\begin{pmatrix} 1 \end{pmatrix}$, so $\sigma(M^8) = 1$. As M is smooth, we can apply **Prop. 2.6** to obtain

$$1 = \frac{1}{45}(7p_2 - p_1^2)[M^8] \tag{1}$$

where the p_i are the pontrjagin classes of the tangent bundle of M^8.

Also, **Claim 3.3** says that $p_1^2[M^8] = k^2u^2[M^8] = k^2$. Plugging into equation (1) gives:

$$1 = \frac{1}{45}(7p_2[M^8] - k^2) \;\Rightarrow\; p_2[M^8] = \frac{1}{7}(45 + k^2)$$

But if $k = 6$ ($\equiv 2 \pmod 4$), we discover that $p_2[M^8] \notin \mathbb{Z}$. This is a contradiction! Thus $S(E)$ is homeomorphic, but *not* diffeomorphic, to S^7.

4. The Construction

Here we'll actually construct the bundle E from **Claim 3.2** using two other bundles over S^4, E_1 and E_2.

Let $E_1 = TS^4$, the tangent bundle of S^4, and define the second bundle in the following way: note that $S^4 = \mathbb{H}P^1$, and let E_2 be the canonical quaternionic line bundle (which we can view as a 4-dimensional real vector bundle).

Notice first that $e(E_1) = 2u$, where u is a generator of $H^4(S^4;\mathbb{Z})$. This follows from the following formula for smooth closed orientable manifolds, which gives the euler class its name, but which we don't prove here (see [MS]):

$$\langle e(TM); [M]\rangle = \chi(M)$$

where $\chi(M)$ is the euler characteristic of M. In our case $\chi(S^4) = 2$, from which the result follows. Also recall that, as we showed earlier, $p_1(S^4) = 0$.

So what's up with our second bundle?

Claim 4.1. *$e(E_2) = \tilde{u}$ and $p_1(E_2) = 2\tilde{u}$, where $\tilde{u}$ is a generator of $H^4(S^4;\mathbb{Z})$.*

Proof. Notice that this bundle has fiber $\mathbb{H}$, so the associated sphere bundle has fiber the unit quaternions, i.e., S^3. It follows that this sphere bundle is really just the Hopf fibration $S^3 \to S^7 \to S^4$. So now use the Gysin sequence from the proof of **Prop 3.1**, but in reverse: this time the fact that $E - s_0 \simeq S^7$ tells us that $\smile e : H^0(S^4;\mathbb{Z}) \to H^4(S^4;\mathbb{Z})$ is an isomorphism. Thus e must be a generator, say $\tilde{u}$.

Now recall that the pontrjagin classes of a vector bundle E are defined in terms of the chern classes of the complexified bundle: $p_i(E) := (-1)^i c_{2i}(E \otimes \mathbb{C})$. It's a straightforward calculation that, for a bundle E that's already complex, $E \otimes \mathbb{C} \cong$

$E \oplus \bar{E}$, where $\bar{E}$ has the opposite orientation. Thus, since E_2 natually inherits a complex structure, we have

$$\begin{aligned} p_1(E_2) &= (-1)^1 c_2(E_2 \otimes \mathbb{C}) \\ &= -c_2(E_2 \oplus \bar{E_2}) \\ &= -(c_2(E_2) + c_2(\bar{E_2})) \\ &= -2c_2(E_2) \end{aligned}$$

where the penultimate equality is given by the Whitney sum formula for chern classes, and the last comes from the formula $c_i(\bar{E}) = (-1)^i c_i(E)$. But recall that the top chern class (in our case c_2) is always equal to the euler class, so $p_1(E_2) = -2e(E_2) = -2\tilde{u}$. □

So we can finally construct our bundle E from E_1 and E_2. Recall that in the theory of classifying spaces any oriented vector bundle over S^4 can be identified with a homotopy class of maps $S^4 \to BSO(4)$. So we can view $E_1, E_2 \in \pi_4(BSO(4))$. This is just an abelian group, so for integers a and b, $aE_1 + bE_2 \in \pi_4(BSO(4))$ represents another vector bundle over S^4. Define E to be this vector bundle.

Claim 4.2. *$e(E) = 2a \cdot u + b \cdot \tilde{u}$ and $p_1(E) = -2b \cdot \tilde{u}$*

Proof. Notice that the map $aE_1 + bE_2$ factors in the following way:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \tilde{E} \arrow[r] \arrow[d] & ESO(4) \arrow[d] \\ S^4 \arrow[r, "h"] & \bigvee^{a+b} S^4 \arrow[r, "f"] & BSO(4) \end{tikzcd}$$

where h is the (based) pinch map, and $\tilde{E}$ is the pullback bundle via the map f. To see what the induced maps on cohomology do, consider the following composition:

$$S^4 \overset{i}{\hookrightarrow} \bigvee^{a+b} S^4 \xrightarrow{f} BSO(4)$$

where i is just the inclusion of S^4 (equipped with bundle, say, E_j) as one of the $a+b$ petals corresponding to E_j in the wedge. Thus by naturality of the euler class

$$e(E_j) = (f \circ i)^*(e(BSO(4))) = i^*(e(\tilde{E}))$$

Since the induced map $i^* : H^4(\bigvee^{a+b} S^4; \mathbb{Z}) = \bigoplus^{a+b} H^4(S^4; \mathbb{Z}) \to H^4(S^4; \mathbb{Z})$ is just projection onto the appropriate summand, it follows that

$$e(\tilde{E}) = (\underbrace{e(E_1), ..., e(E_1)}_{a \text{ times}}, \underbrace{e(E_2), ..., e(E_2)}_{b \text{ times}})$$

It's also easy to check that $h^* : \bigoplus^{a+b} H^4(S^4; \mathbb{Z}) \to H^4(S^4; \mathbb{Z})$, induced by the pinch map, just sums the coordinates in an $(a+b)$-tuple. So

$$h^*(\underbrace{e(E_1), ..., e(E_1)}_{a \text{ times}}, \underbrace{e(E_2), ..., e(E_2)}_{b \text{ times}}) = a \cdot e(E_1) + b \cdot e(E_2) = 2a \cdot u + b \cdot \tilde{u}$$

Naturality says this is precisely $e(E)$.

One can use an identical argument for the first pontrjagin class, for which naturality also holds. □

Now we have two possible cases. If $u = \tilde{u}$, then $e(E) = (2a + b)u$ and $p_1(E) = -2bu$. Choosing $2a + b = 1$, we get $p_1(E) = (4a - 2)u = (4a - 2)e(E)$

If $u = -\tilde{u}$, then $e(E) = (2a - b)u$ and $p_1(E) = 2bu$. This time $2a - b = 1$ also gives us $p_1(E) = (4a - 2)e(E)$. In either case $e(E)$ is a generator of $H^4(S^4; \mathbb{Z})$, and a proper choice of a yields $(4a - 2) = k \equiv 2 (\bmod 4)$ for any such k. This completes the construction.

It's worth briefly noting another construction of E that has a very nice geometric interpretation, though perhaps less straightforward algebraic computations. We'll create the bundle over S^4 by simply gluing together two trivial bundles:

$$E := (D^4 \times \mathbb{H}) \cup_g (D^4 \times \mathbb{H})$$

where g glues along the boundaries via

$$\begin{aligned} g : S^3 \times \mathbb{H} &\rightarrow S^3 \times \mathbb{H} \\ (x, y) &\rightarrow (x, x^t y x^s) \end{aligned}$$

for $t, s \in \mathbb{Z}$, and viewing $S^3 \subset \mathbb{H}$, the group of unit vectors. This construction is completely analogous to the Möbius bundle over S^1. Similarly, the associated guys

$$D(E) = (D^4 \times D^4) \cup_g (D^4 \times D^4)$$

$$S(E) = (D^4 \times S^3) \cup_g (D^4 \times S^3)$$

are analogous to the Möbius strip and the double cover of S^1. Here we'll find that $S(E)$ is homeomorphic to S^7 whenever $t + s = 1$, and setting $s = 1 - t$ we get a one-parameter family of topological 7-spheres. But letting $t = 2$ will tell us that $p_1^2[M^8] = 36$, giving us the same situation we saw in Section 3. See [Kr] for details.

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