Random Walk Algorithms: Homework 5

Consider a random walk on $\mathbb{Z}/N\mathbb{Z}$, $N \in \mathbb{N}$, with a "sink" at $N-1$, *i.e.*, with a transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/4 & & & & \\ 1/4 & 1/2 & 1/4 & & & \\ & 1/4 & 1/2 & & & \\ & & 1/4 & \ddots & 1/4 & \\ & & & & 1/2 & \\ 1/4 & & & & 1/4 & 1 \end{pmatrix}.$$

1. Write code to start at a uniformly random state and evolve according to this random walk. For $N \in \{10, 20, 30\}$, show histograms of the position after $T \in \{100, 500, 1000\}$ steps.

2. a. Find a stationary distribution for the stochastic process defined by P.

   b. Let $\theta = \pi/N$. Show that the vector v with

   $$v_s = \begin{cases} -\sin\big((s+1)\theta\big) & \text{if } s \in \{0, 1, \ldots, N-2\} \\ \sin\theta/(1-\cos\theta) & \text{if } s = N-1 \end{cases}$$

   is an eigenvector of P and find its eigenvalue.

   c. Use your answers for questions 2.a. and 2.b. to explain your results in problem 1, specifically how close each histogram is to the stationary distribution.