Random Walk Algorithms: Homework 8

The quantum algorithm to solve the SEARCH problem for $w \in \{1, \ldots, N\}$ that we derived in class consists of repeatedly applying the unitary transformation

$$(2uu^\dagger - I)(I - 2\hat{e}_w \hat{e}_w^\dagger),$$

where $u = (1, \ldots, 1)/\sqrt{N}$ and $\hat{e}_w$ is the vector with components $(\hat{e}_w)_x = \delta_{xw}$, to the initial vector u. Let

$$\psi_t = \big((2uu^\dagger - I)(I - 2\hat{e}_w \hat{e}_w^\dagger)\big)^t u.$$

1. For $N = 1024$, plot $|\hat{e}_w^\dagger \psi_t|^2$ for $t \in \{0, 1, \ldots, 50\}$. Which is the first value of t at which this probability is a local maximum (*i.e.*, greater than at $t-1$ and at $t+1$)?

2. Let $W \subset \{1, \ldots, N\}$ have k elements. Modify the quantum algorithm above by replacing $I - 2\hat{e}_w \hat{e}_w^\dagger$ with
$$I - 2 \sum_{w \in W} \hat{e}_w \hat{e}_w^\dagger.$$
This is an algorithm for finding any one of the elements in W. For $N = 1024$, plot the success probability
$$\sum_{w \in W} |\hat{e}_w^\dagger \psi_t|^2,$$
for $t \in \{0, 1, \ldots, 50\}$ when $k = 4$ and $k = 16$. In each case, which is the first value of t at which this probability is a local maximum (*i.e.*, greater than at $t-1$ and at $t+1$)?

3. [Bonus] Let T_k be the first value of t at which the success probability for finding one of the k elements in $W \subset \{1, \ldots, N\}$ is a local maximum. For example, we proved in class that
$$T_1 = \left\lfloor \frac{\pi}{4}\sqrt{N} - \frac{1}{2} \right\rceil.$$
Show that $T_k = O(\sqrt{N/k})$ as $N \to \infty$.