168A FINAL EXAM

1. [15 points] Define a random walk on $\mathbb{N} = \{0, 1, \ldots\}$ by

$$\Pr(X_{t+1} = x_{t+1} \mid X_t = x_t) = \begin{cases} 1/2 & \text{if } x_{t+1} = x_t; \\ 1/2 & \text{if } x_{t+1} = 1 \text{ and } x_t = 0; \\ 1/4 & \text{if } x_t \neq 0 \text{ and } |x_{t+1} - x_t| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

   Show that this random walk has no stationary distribution.

2. [20 points] Suppose there is a graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, but you do not know either V or E. All you know to begin is a single $v_0 \in V$, and also that V is finite and every vertex in G is connected to no more than D other vertices. You can call a subroutine, nn(), that, when you pass it a vertex, returns a list of all the vertices to which that vertex is connected by an edge in G. Write pseudocode for a random walk algorithm that returns a vertex from the component of G in which v_0 lies, approximately uniformly at random, if you run it long enough. You do not have to specify how long is long enough. Hint: Define a transition probability matrix that has the uniform distribution as a stationary distribution.

3. Two people are placed on an unlabeled number line, at different integers. At each timestep, from position x each can step to position $x-1$ or $x+1$, or remain at position x. Suppose each person walks randomly, with probabilities p of stepping to the right, p of stepping to the left, and $1-2p$ of staying at the same position, where $0 \leq p \leq 1/2$.

   a. [2 points] If they start on adjacent numbers, what is the probability they will collide in the first timestep?

   b. [3 points] What value of p maximizes the probability you found in (a)?

   c. [10 points] Suppose the goal of the people is to minimize the expected time until they find one another, T, but they don't know how far apart they are, nor in which direction the other person is. What should they choose for p? Please justify your answer. Hint: Think about the solution to the diffusion equation.

   d. [5 points] Now suppose one person wants to find the other, but the second doesn't want to be found, *i.e.*, the first wants to minimize $\mathsf{E}[T]$ while the second wants to maximize it. If they can choose different random walk parameters, what should each of them choose? Again, please justify your answer.

   e. [5 points] Are there better algorithms than a random walk for the players to follow in (c) or in (d)? If so, what are they and why are they better?

4. Let $\{\hat{e}_0, \hat{e}_1\}$ be an orthornormal basis for $\mathbb{C}^2$. Then any quantum state $\psi \in \mathbb{C}^2$ can be written as $\psi = e^{i\alpha}(\cos(\frac{\theta}{2})\hat{e}_0 + e^{i\phi}\sin(\frac{\theta}{2})\hat{e}_1)$, where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Ignoring the overall factor of $e^{i\alpha}$, ψ can be identified with a point on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, using spherical coordinates in which θ measures the angle from $\hat{z}$ and ϕ is the angle of the projection into the x-y plane from $\hat{x}$. Any unitary matrix multiplying quantum states in $\mathbb{C}^2$ acts on $\mathbb{S}^2$ by moving the corresponding points.

   a. [5 points] Describe the action of $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{S}^2$.

   b. [5 points] Describe the action of $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{S}^2$.

   c. [5 points] Describe the action of $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ on $\mathbb{S}^2$.

5. Let $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 \\ \sqrt{N-1} \end{pmatrix}$, for $N \in \mathbb{N}$.

   a. [5 points] Find the entries of the 2×2 matrix $I - 2ww^\dagger$.

   b. [5 points] Let $\sin\alpha = 1/\sqrt{N}$. Write u in terms of α.

   c. [5 points] Find the entries of the 2×2 matrix $2uu^\dagger - I$, in terms of α.

   d. [5 points] Show that $(2uu^\dagger - I)(I - 2ww^\dagger)$ is a matrix that rotates vectors by angle 2α.

   e. [5 points] Draw the same sphere as in problem 4, and mark w and u on it. Mark the points to which u is mapped by repeated application of the matrix in (d).