## 168A PRACTICE FINAL

1. Suppose le Petit Prince lands on a soccer ball (truncated icosahedron) planet and randomly walks from face to face, with equal probabilities of stepping to each adjacent face from the one he is currently on. If he stops after many steps, what is the probability that he is on a pentagon? Hint: There are 12 pentagons and 20 hexagons.

2. Two bugs start at opposite corners of a square. During each minute they each walk along an edge, chosen uniformly at random, to an adjacent corner. If they run into each other on an edge they stop, and if they end at the same corner, they stop. Let $T$ be the minute during which they stop.
a. Describe this situation as a discrete time Markov chain: specify a set of states for the system and write down the one step probability transition matrix.
b. What is $\mathrm{E}[T]$ ?
c. What would $\mathrm{E}[T]$ be if the bugs start on adjacent vertices?
3. Let $\left\{\hat{e}_{0}, \hat{e}_{1}\right\}$ be an orthornormal basis for $\mathbb{R}^{2}$. Then any probability distribution over two states $\{0,1\}$ can be written as $v=p \hat{e}_{0}+(1-p) \hat{e}_{1}$, where $0 \leq p \leq 1 . v$ can be identified with the point $2 p-1 \in[-1,1] \subset \mathbb{R}$.
a. Draw $[-1,1] \subset \mathbb{R}$ and mark $\hat{e}_{0}$ and $\hat{e}_{1}$ on it.
b. Label the point $\left(\hat{e}_{0}+\hat{e}_{1}\right) / 2$ on $[-1,1] \subset \mathbb{R}$.
c. For $0 \leq q \leq 1$, describe the action of the transition probability matrix $\left(\begin{array}{cc}1-q & q \\ q & 1-q\end{array}\right)$ on $[-1,1] \subset \mathbb{R}$ as a function $f:[-1,1] \rightarrow[-1,1]$.
d. Prove that there exists an $x \in[-1,1]$ such that $f(x)=x$, where $f$ is the function you found in (c).

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4. Let $\left\{\hat{e}_{0}, \hat{e}_{1}\right\}$ be an orthornormal basis for $\mathbb{C}^{2}$. Then any quantum state $\psi \in \mathbb{C}^{2}$ can be written as $\psi=e^{i \alpha}\left(\cos \left(\frac{\theta}{2}\right) \hat{e}_{0}+e^{i \phi} \sin \left(\frac{\theta}{2}\right) \hat{e}_{1}\right)$, where $0 \leq \theta \leq \pi$ and $0 \leq \phi<2 \pi$. Ignoring the overall factor of $e^{i \alpha}, \psi$ can be identified with a point on the unit sphere $\mathbb{S}^{2} \subset \mathbb{R}^{3}$, using spherical coordinates in which $\theta$ measures the angle from $\hat{z}$ and $\phi$ is the angle of the projection into the $x-y$ plane from $\hat{x}$.
a. Draw $\mathbb{S}^{2} \subset \mathbb{R}^{3}$ and mark $\hat{e}_{0}$ and $\hat{e}_{1}$ on it.
b. Label the points $\left(\hat{e}_{0}+\hat{e}_{1}\right) / \sqrt{2}$ and $\left(\hat{e}_{0}-\hat{e}_{1}\right) / \sqrt{2}$ on $\mathbb{S}^{2}$.
c. There are two unit vectors that are perpendicular in $\mathbb{R}^{3}$ to all four vectors you drew in parts (a) and (b). Which quantum states are they?
5. The quantum walk we derived on $\mathbb{Z} / N \mathbb{Z}$ has states in $\mathbb{C}^{2 N}$. Writing a basis for this vector space as $\left\{\hat{e}_{x, \alpha} \mid x \in \mathbb{Z} / N \mathbb{Z}, \alpha \in\{-1,+1\}\right\}$, the unitary evolution acts by

$$
\begin{aligned}
& \hat{e}_{x,-1} \mapsto \cos \theta \hat{e}_{x-1,-1}+i \sin \theta \hat{e}_{x-1,+1} \\
& \hat{e}_{x,+1} \mapsto i \sin \theta \hat{e}_{x+1,-1}+\cos \theta \hat{e}_{x+1,+1} .
\end{aligned}
$$

If $\psi_{0}=\left(\hat{e}_{0,-1}+\hat{e}_{0,+1}\right) / \sqrt{2}$, what is $\psi_{2}$ ?

