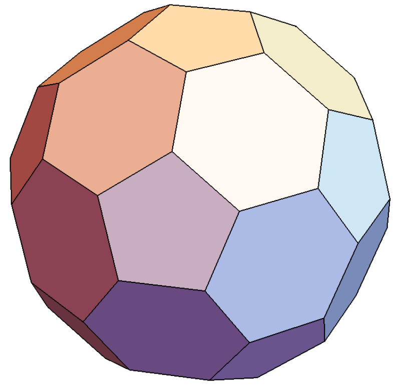
168A PRACTICE FINAL SOLUTIONS



1. Suppose *le Petit Prince* lands on a soccer ball (truncated icosahedron) planet and randomly walks from face to face, with equal probabilities of stepping to each adjacent face from the one he is currently on. If he stops after many steps, what is the probability that he is on a pentagon? Hint: There are 12 pentagons and 20 hexagons.

   The stationary distribution of the random walk on the dual graph has probabilities proportionally to the degrees of the vertices, so the ratio of the probability of being on a pentagon to being on a hexagon is $5 \cdot 12/6 \cdot 20 = 1/2$, so the probability of being on a pentagon is $1/3$.

2. Two bugs start at opposite corners of a square. During each minute they each walk along an edge, chosen uniformly at random, to an *adjacent* corner. If they run into each other on an edge they stop, and if they end at the same corner, they stop. Let T be the minute during which they stop.

   a. Describe this situation as a discrete time Markov chain: specify a set of states for the system and write down the one step probability transition matrix.

   Let X_t = how far apart the bugs are at time t, counted by numbers of edges one must traverse to reach the other $\in \{0, 1, 2\}$. Then

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 1/4 & 1/2 \\ 0 & 3/4 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \end{array}.$$

   b. What is $\mathsf{E}[T]$?

   Let $\tau_i = \mathsf{E}[T | X_0 = i]$. Then we have

$$\begin{aligned} \tau_0 &= 0 \\ \tau_1 &= 1 + \tau_1 \cdot \frac{3}{4} \quad &\Rightarrow \quad \tau_1 = 4 \\ \tau_2 &= 1 + \tau_2 \cdot \frac{1}{2} \quad &\Rightarrow \quad \tau_2 = 2 \end{aligned}$$

   so since the bugs start at opposite corners, $\mathsf{E}[T] = 2$.

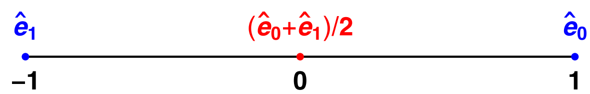
   c. What would $\mathsf{E}[T]$ be if the bugs start on adjacent vertices?

   As calculated above, in this case, $\mathsf{E}[T] = \tau_1 = 4$.

3. Let $\{\hat{e}_0, \hat{e}_1\}$ be an orthornormal basis for $\mathbb{R}^2$. Then any probability distribution over two states $\{0,1\}$ can be written as $v = p\hat{e}_0 + (1-p)\hat{e}_1$, where $0 \le p \le 1$. v can be identified with the point $2p-1 \in [-1,1] \subset \mathbb{R}$.

   a. Draw $[-1,1] \subset \mathbb{R}$ and mark $\hat{e}_0$ and $\hat{e}_1$ on it.

   b. Label the point $(\hat{e}_0 + \hat{e}_1)/2$ on $[-1,1] \subset \mathbb{R}$.



   c. For $0 \le q \le 1$, describe the action of the transition probability matrix $\begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix}$ on $[-1,1] \subset \mathbb{R}$ as a function $f : [-1,1] \to [-1,1]$.

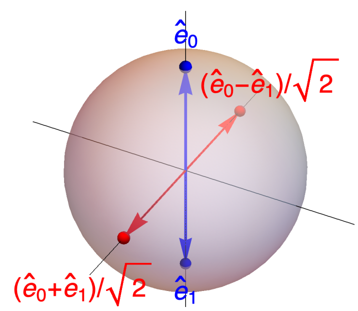
$$\begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} (1-q)p + q(1-p) \\ 1 - \text{ above} \end{pmatrix},$$

   so $p \mapsto p + q - 2pq$. The coordinate $x \in [-1,1]$ to which $(p, 1-p)$ is mapped is $2p-1$; thus $2p-1 \mapsto 2(p+q-2pq) - 1 = 2p - 1 - 2q(2p-1)$, so $x \mapsto (1-2q)x$, *i.e.*, $f(x) = (1-2q)x$.

   d. Prove that there exists an $x \in [-1,1]$ such that $f(x) = x$, where f is the function you found in (c).

   $f(0) = (1-2q) \cdot 0 = 0$. Alternatively, since any stochastic matrix has an eigenvalue 1 eigenvector, there must always be some x which is fixed by the corresponding transformation on the interval $[-1,1]$.

4. Let $\{\hat{e}_0, \hat{e}_1\}$ be an orthornormal basis for $\mathbb{C}^2$. Then any quantum state $\psi \in \mathbb{C}^2$ can be written as $\psi = e^{i\alpha}(\cos(\frac{\theta}{2})\hat{e}_0 + e^{i\phi}\sin(\frac{\theta}{2})\hat{e}_1)$, where $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. Ignoring the overall factor of $e^{i\alpha}$, ψ can be identified with a point on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, using spherical coordinates in which θ measures the angle from $\hat{z}$ and ϕ is the angle of the projection into the x-y plane from $\hat{x}$.



   a. Draw $\mathbb{S}^2 \subset \mathbb{R}^3$ and mark $\hat{e}_0$ and $\hat{e}_1$ on it.

   b. Label the points $(\hat{e}_0 + \hat{e}_1)/\sqrt{2}$ and $(\hat{e}_0 - \hat{e}_1)/\sqrt{2}$ on $\mathbb{S}^2$.

   c. There are two unit vectors that are perpendicular in $\mathbb{R}^3$ to all four vectors you drew in parts (a) and (b). Which quantum states are they?

   $(\hat{e}_0 + i\hat{e}_1)/\sqrt{2}$ and $(\hat{e}_0 - i\hat{e}_1)/\sqrt{2}$.

5. The quantum walk we derived on $\mathbb{Z}/N\mathbb{Z}$ has states in $\mathbb{C}^{2N}$. Writing a basis for this vector space as $\left\{\hat{e}_{x,\alpha} \mid x \in \mathbb{Z}/N\mathbb{Z}, \alpha \in \{-1,+1\}\right\}$, the unitary evolution acts by

$$\begin{aligned} \hat{e}_{x,-1} &\mapsto \cos\theta\,\hat{e}_{x-1,-1} + i\sin\theta\,\hat{e}_{x-1,+1} \\ \hat{e}_{x,+1} &\mapsto i\sin\theta\,\hat{e}_{x+1,-1} + \cos\theta\,\hat{e}_{x+1,+1} \end{aligned}.$$

If $\psi_0 = (\hat{e}_{0,-1} + \hat{e}_{0,+1})/\sqrt{2}$, what is ψ_2?

To compress the notation, let $c = \cos\theta$ and $s = \sin\theta$, and write $|x,\pm\rangle = \hat{e}_{x,\pm 1}$. Then

$$\psi_2 = \frac{1}{\sqrt{2}}\Big(c^2|-2,-\rangle + isc|-2,+\rangle + (isc - s^2)\big(|0,-\rangle + |0,+\rangle\big) + isc|2,-\rangle + c^2|2,+\rangle\Big).$$