168A FINAL EXAM SOLUTIONS

1. [15 points] Define a random walk on $\mathbb{N} = \{0, 1, \ldots\}$ by

$$\Pr(X_{t+1} = x_{t+1} \mid X_t = x_t) = \begin{cases} 1/2 & \text{if } x_{t+1} = x_t; \\ 1/2 & \text{if } x_{t+1} = 1 \text{ and } x_t = 0; \\ 1/4 & \text{if } x_t \neq 0 \text{ and } |x_{t+1} - x_t| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Show that this random walk has no stationary distribution.

Suppose this random walk has stationary distribution $(p_0, p_1, \ldots)$. Then

$$\begin{aligned} \frac{1}{2}p_0 &= \frac{1}{4}p_1 \\ \frac{1}{2}p_1 &= \frac{1}{2}p_0 + \frac{1}{4}p_2 \\ \frac{1}{2}p_2 &= \frac{1}{4}p_1 + \frac{1}{4}p_3 \\ &\vdots \end{aligned}$$

The first equation implies $p_1 = 2p_0$; then the second implies $p_2 = 2p_0$; then the third implies $p_3 = 2p_0$; .... Thus the stationary distribution would be $p_0(1, 2, 2, \ldots)$. But there is no value of p_0 that makes these probabilities sum to 1, which contradicts them being a probability distribution, so there is no stationary distribution.

2. [20 points] Suppose there is a graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, but you do not know either V or E. All you know to begin is a single $v_0 \in V$, and also that V is finite and every vertex in G is connected to no more than D other vertices. You can call a subroutine, nn(), that, when you pass it a vertex, returns a list of all the vertices to which that vertex is connected by an edge in G. Write pseudocode for a random walk algorithm that returns a vertex from the component of G in which v_0 lies, approximately uniformly at random, if you run it long enough. You do not have to specify how long is long enough. Hint: Define a transition probability matrix that has the uniform distribution as a stationary distribution.

A random walk on a finite graph has a stationary distribution in which the probability of being at each vertex is proportional to the degree of that vertex. Given a graph in which not all the vertices have the same degree, we can add self-loops to make all the vertices have the same effective degree. More precisely, if the degree of vertex v is d_v, and the largest degree is D, then $D - d_v$ self-loops, with equal probability to go around each as to take each of the d_v edges incident at v, make v have effective degree D. The following pseudocode describes a random walk on this modified graph.

input: $v_0 \in V$; $D \in \mathbb{N}$.
output: v in the component of V containing v_0.

$N \leftarrow 0$
$v \leftarrow v_0$
while $N < 10^6$ # 10^6 is an arbitrarily chosen large number
  $neighbors \leftarrow \text{nn}(v)$
  $d_v \leftarrow |neighbors|$
  $i \leftarrow \text{random}(D)$ # a random integer in the set $\{1, \ldots, D\}$
  if $i \leq d_v$, $v \leftarrow neighbors_i$
  $N \leftarrow N + 1$
return v

There are other possible algorithms, *e.g.*, one might construct a list of vertices, and deterministically check each for its neighbors, adding them to the list, until the process terminates.

3. Two people are placed on an unlabeled number line, at different integers. At each timestep, from position x each can step to position $x-1$ or $x+1$, or remain at position x. Suppose each person walks randomly, with probabilities p of stepping to the right, p of stepping to the left, and $1-2p$ of staying at the same position, where $0 \leq p \leq 1/2$.

   a. [2 points] If they start on adjacent numbers, what is the probability they will collide in the first timestep?

   $p^2 + 2p(1-2p) = 2p - 3p^2$

   b. [3 points] What value of p maximizes the probability you found in (a)?

   Taking the derivative with respect to p gives $2 - 6p = 0 \Rightarrow p = 1/3$.

   c. [10 points] Suppose the goal of the people is to minimize the expected time until they find one another, T, but they don't know how far apart they are, nor in which direction the other person is. What should they choose for p? Please justify your answer. Hint: Think about the solution to the diffusion equation.

   If $p = 0$, the expected time until they find one another is infinite, but as long as $p > 0$, it is finite: Suppose they start distance d apart. Since the probability distribution of the change in d is symmetric around 0, there is equal probability (namely 1/2) of the distance going first to 0 or first to $2d$. If it goes to 0, the game is over; if it goes to $2d$ they continue, and now there is probability 1/2 of the distance going first to 0 or first to $4d$. Thus the probability of them never meeting is $1-1/2-1/4-1/8-\cdots = 0$, which proves that the expected time until they meet is finite. This time is monotonically decreasing as p increases, so they should choose $p = 1/2$.

   d. [5 points] Now suppose one person wants to find the other, but the second doesn't want to be found, *i.e.*, the first wants to minimize $\mathsf{E}[T]$ while the second wants to maximize it. If they can choose different random walk parameters, what should each of them choose? Again, please justify your answer.

Even if the random walk parameters are different for the two players, the probability distribution for the change in the distance between them is symmetric. Thus the expected time to meet is finite as long as both of the players don't choose $p = 0$. Since the first player wants to find the second, the first won't choose $p = 0$, so the expected time to meet is finite. The expected time will decrease as the variance of the probability distribution for the change in distance increases, so the first player will choose $p = 1/2$, while the second player will choose $p = 0$.

e. [5 points] Are there better algorithms than a random walk for the players to follow in (c) or in (d)? If so, what are they and why are they better?

In (d) it would be better for the second player to pick one direction at random and always step in that direction. If s/he guesses wrong, s/he'll meet the first player sooner than if s/he stayed still, but if s/he guesses right, they'll *never* meet, so the expected time to meet is infinite.

4. Let $\{\hat{e}_0, \hat{e}_1\}$ be an orthornormal basis for $\mathbb{C}^2$. Then any quantum state $\psi \in \mathbb{C}^2$ can be written as $\psi = e^{i\alpha}(\cos(\frac{\theta}{2})\hat{e}_0 + e^{i\phi}\sin(\frac{\theta}{2})\hat{e}_1)$, where $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. Ignoring the overall factor of $e^{i\alpha}$, ψ can be identified with a point on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, using spherical coordinates in which θ measures the angle from $\hat{z}$ and ϕ is the angle of the projection into the x-y plane from $\hat{x}$. Any unitary matrix multiplying quantum states in $\mathbb{C}^2$ acts on $\mathbb{S}^2$ by moving the corresponding points.

a. [5 points] Describe the action of $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{S}^2$.

$$X\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} = e^{i\phi}\begin{pmatrix} \sin\frac{\theta}{2} \\ e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix} = e^{i\phi}\begin{pmatrix} \cos\frac{\pi-\theta}{2} \\ e^{-i\phi}\sin\frac{\pi-\theta}{2} \end{pmatrix},$$

so X maps θ to $\pi - \theta$ and ϕ to $-\phi$. This is a rotation by π around $\hat{x}$.

b. [5 points] Describe the action of $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{S}^2$.

$$Z\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i(\phi+\pi)}\sin\frac{\theta}{2} \end{pmatrix},$$

so Z maps θ to θ and ϕ to $\phi + \pi$. This is a rotation by π around $\hat{z}$.

c. [5 points] Describe the action of $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ on $\mathbb{S}^2$.

$$H\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} - e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix},$$

which, although it can be, is a little more complicated to convert into the form $e^{i\alpha}(\cos(\frac{\theta'}{2})\hat{e}_0 + e^{i\phi'}\sin(\frac{\theta'}{2})\hat{e}_1)$. Instead, note that H takes $\hat{e}_0$ to $(\hat{e}_0 + \hat{e}_1)/\sqrt{2}$, and $\hat{e}_1$ to $(\hat{e}_0 - \hat{e}_1)/\sqrt{2}$, and it also takes $(\hat{e}_0 + i\hat{e}_1)/\sqrt{2}$ to $(\hat{e}_0 - i\hat{e}_1)/\sqrt{2}$; extrapolating from parts (a) and (b) we might guess that this means it is a rotation of $\mathbb{S}^2$, by π around the vector $(\hat{x}+\hat{z})/\sqrt{2}$. To confirm this, check that this vector is left invariant by H:

$$H\begin{pmatrix} c \\ s \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} c+s \\ c-s \end{pmatrix},$$

where $c = \cos(\pi/8)$ and $s = \sin(\pi/8)$. For this vector to be unchanged, we must have $c+s = \sqrt{2}c$ and $c-s = \sqrt{2}s$. Multiplying these two equations gives $(c+s)(c-s) = c^2 - s^2 = 2cs$, which is $\cos(2\cdot\frac{\pi}{8}) = \sin(2\cdot\frac{\pi}{8})$, which is true.

5. Let $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 \\ \sqrt{N-1} \end{pmatrix}$, for $N \in \mathbb{N}$.

a. [5 points] Find the entries of the 2×2 matrix $I - 2ww^\dagger$.

$$I - 2ww^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

b. [5 points] Let $\sin\alpha = 1/\sqrt{N}$. Write u in terms of α.

$(\sin\alpha, \cos\alpha)$

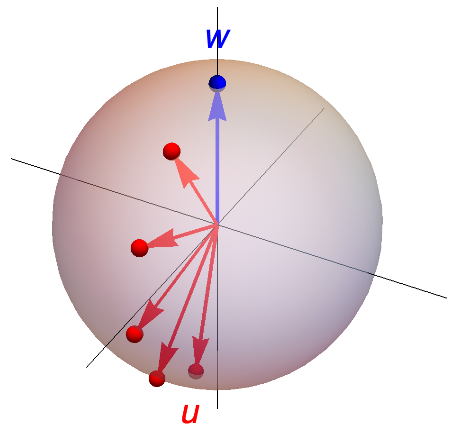
c. [5 points] Find the entries of the 2×2 matrix $2uu^\dagger - I$, in terms of α.

$$2uu^\dagger - I = 2\begin{pmatrix} \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2\sin^2\alpha - 1 & 2\sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & 2\cos^2\alpha - 1 \end{pmatrix}$$

d. [5 points] Show that $(2uu^\dagger - I)(I - 2ww^\dagger)$ is a matrix that rotates vectors by angle 2α.

$(2uu^\dagger - I)(I - 2ww^\dagger) = \begin{pmatrix} 1-2\sin^2\alpha & 2\sin\alpha\cos\alpha \\ -2\sin\alpha\cos\alpha & 2\cos^2\alpha - 1 \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix}$. We can recognize this immediately as a rotation matrix through angle 2α, or we can demonstrate that it is by multiplying it times a general vector $(\sin\beta, \cos\beta)$ to get the vector $(\sin(\beta+2\alpha), \cos(\beta+2\alpha))$.

e. [5 points] Draw the same sphere as in problem 4, and mark w and u on it. Mark the points to which u is mapped by repeated application of the matrix in (d).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