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Random Walk Algorithms: Lecture 11

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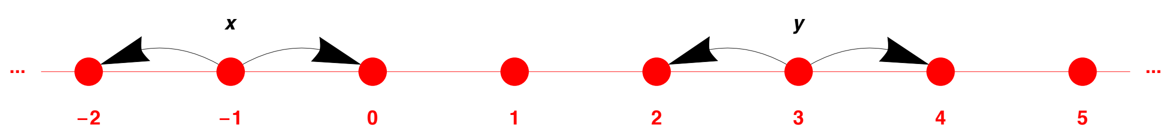
In which multiple particles diffusing in 1D is modeled by multiple independent random walks on $\mathbb{Z}$, which are identified with a single random walk in $\mathbb{Z}^d$.

Physics

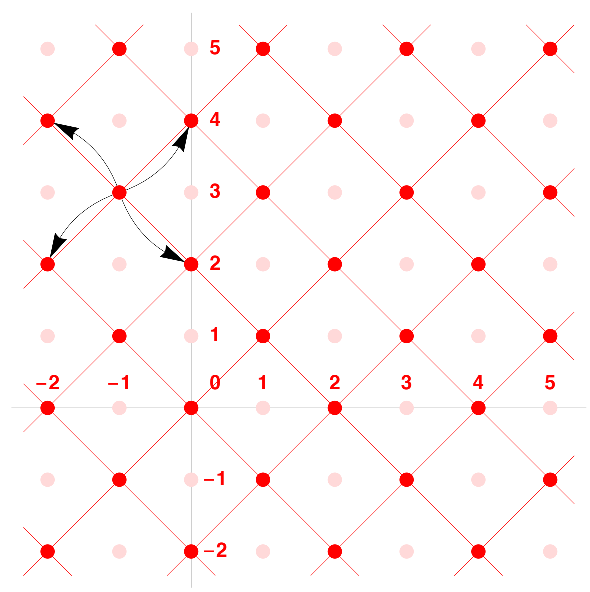
The basis for the connection between the heat or diffusion equation and random walks is the physics of diffusion—a process wherein many particles spread out as they bounce around in some medium. Each is acting essentially like an independent random walk, so the density of particles is equivalent to the histogram of many samples from a random walk, *i.e.*, the density should evolve like the probability distribution function.

Two particles

Consider the simple case of two particles randomly walking, independently, on $\mathbb{Z}$:

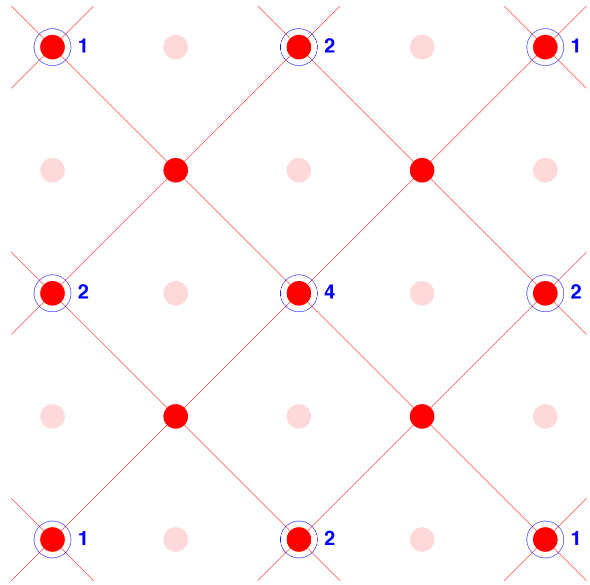


We could also describe this situation by using the positions of the two particles in one dimension as coordinates of a single particle in two dimensions, as shown in the adjacent figure. Now there are 4 states to which there is a nonzero (and equal) transition probability from any initial state (x, y): $\{(x\pm1, y\pm1)\}$. That is, it is a random walk on the "diamond" graph shown, with transition probabilities of $1/4$ along each edge.



In general, we can define a random walk on any graph. Let A be the *adjacency matrix* for the graph, *i.e.*, $A_{ij} = 1$ if vertices i and j are connected by an edge in the graph, and $A_{ij} = 0$ otherwise. If $d_j = \sum_i A_{ij}$ is the number of vertices to which vertx j is connected, the *degree* of vertex j, then a random walk on the state space consisting of the vertices of the graph has transition probability matrix with entries $P_{ij} = A_{ij}/d_j$. Random walks on graphs are the basis for many algorithms.

Returning to the case of the "diamond" graph, consider where the particle can be after two steps. The possible positions are circled in the adjacent figure, assuming the particle started at the central vertex. The probability of arriving at each vertex can be computed by counting the number of two step paths to that vertex, shown in the figure, and then dividing by $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.



More generally, after $2t$ steps, if the particle starts at the origin,

$$
\begin{aligned}
&\Pr(X = x, Y = y) \\
&\qquad = \Pr(X = x) \Pr(Y = y) \\
&\qquad = \frac{1}{2^{2t}} \binom{2t}{t + x/2} \frac{1}{2^{2t}} \binom{2t}{t + y/2}.
\end{aligned}
$$

This can be approximated by a bivariate Gaussian distribution, since as we saw in Lecture 8, the probability of each component can be approximated by a Gaussian. In fact, the argument there that led to the heat kernel solution of the heat equation can be generalized to two (or any number d) of dimensions, in which the latter takes the form:

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u,$$

where the *Laplacian* $\nabla^2 = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$.