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Random Walk Algorithms: Lecture 12

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In which diffusion is used to blur images, and the problem of unblurring them is posed (with some ideological undertones).

Blurring

We can use the transition probabilities for a two dimensional random walk starting at the origin that we computed in Lecture 11 to write the transition probability matrix entries:

$$P^{2t}_{(i+x,j+y),(i,j)} = \frac{1}{2^{2t}}\binom{2t}{t+x/2}\frac{1}{2^{2t}}\binom{2t}{t+y/2}.$$

Notice that this has the effect of averaging the probability at state (i,j) with the probabilities of the states in the square with side length $2t+1$ around it, weighted according to this bivariate binomial distribution.

This suggests an application: A digital image consists of an array of pixel values. In a greyscale image each of these values is stored as one byte, *i.e.*, as a number in the set $\{0,1,\ldots,255\}$ where the number represents the intensity of light at that pixel, so that 0 is black, and 255 is white. Applying P^{2t} to an array of pixel values replaces each by a binomial average of the values of the pixels in the $(2t+1)\times(2t+1)$ square around it. This has the effect of *blurring* the image, which is something people sometimes want to do to preserve privacy. For example, here is a badly parked Bentley Continental GT:

If we wanted to save the owner the embarrassment of being identified through the license plate number, we might blur that part of the photograph. To keep things simple we will convert the photograph to a greyscale image before applying our blurring algorithm, in which $2t$ is chosen large enough to make the result sufficiently blurry:

input: $m \times n$ image I; rectangle R bounded by $m_1 < m_2$, $n_1 < n_2$.
output: image I' with rectangle R blurred.

for $x = -t$ to t, $y = -t$ to t do

$$P_{xy} \leftarrow \frac{1}{2^{2t}}\binom{2t}{t+x/2}\frac{1}{2^{2t}}\binom{2t}{t+y/2}$$

for $i = 1$ to m, $j = 1$ to n do

if $m_1 \le i \le m_2$ and $n_1 \le j \le n_2$, $I'_{ij} \leftarrow \sum_{x,y=-t}^{t} P_{xy} I_{i+x,j+y}$

else $I'_{ij} \leftarrow I_{ij}$

return I'

Here is the result of applying this algorithm with $2t = 300$:

Of course, some ultra-luxury car owners feel entitled to park badly,* without risking a license plate compromising their privacy—here's a Rolls Royce Wraith, parked at UC San Diego:

* Although, to be a "Fair Witness" [1], I didn't check how many parking permits were displayed in the windshield; it could have been two (but my bet is on zero).

Unblurring?

A natural question now is whether blurring can be undone, *i.e.*, is it invertible? In one dimension, consider the random walk on $\mathbb{Z}/N\mathbb{Z}$, for N even, defined by:

$$\vec{u}_1 = B\vec{u}_0 = \frac{1}{4}\begin{pmatrix} 2 & 1 & & & 1 \\ 1 & 2 & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & 2 & 1 \\ 1 & & & 1 & 2 \end{pmatrix}\vec{u}_0.$$

Given the "blurred" "image" $\vec{u}_1$, the "unblurred" "image" $\vec{u}_0$ could be computed if B^{-1} exist; does it? Notice that $\vec{z} = (1, -1, \ldots, 1, -1)$ is in the nullspace of B, *i.e.*, $B\vec{z}$ is the 0 vector. Thus B is singular and B^{-1} does *not* exist. That is,

$$B\vec{u}_0 = B(\vec{u}_0 + \alpha\vec{z}), \qquad \text{for all } \alpha \in \mathbb{R}.$$

Now, not all $\vec{u}_0 + \alpha\vec{z}$ will be "images" (probability distributions), but as long as 0s don't appear in both even and odd positions, some will be, so the preimage of $\vec{u}_1$ will not be unique.

Diagonalization

Is this all of the "unblurring" ambiguity; that is, does $\vec{z}$ span the nullspace of B? To answer this we must compute the eigenvalues of B. Notice that $B = (X + 2I + X^{-1})/4$, where

$$X = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & & 0 \\ & & & 1 \end{pmatrix} \qquad \text{and} \qquad X^{-1} = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}.$$

Suppose we could diagonalize X, *i.e.*, find its eigenvectors and use them as the columns of a matrix F so that $F^{-1}XF = D$ is a diagonal matrix with the eigenvalues on the diagonal. Notice that $(F^{-1}XF)^{-1} = F^{-1}X^{-1}F$, so F also diagonalizes X^{-1}, and thus B:

$$F^{-1}BF = \frac{1}{4}(F^{-1}XF + 2I + F^{-1}X^{-1}F) = \frac{1}{4}(D + 2I + D^{-1}).$$

In the next lecture we will compute F, and thereby, the eigenvalues of B.

References

[1] R. A. Heinlein, *Stranger in a Strange Land* (New York: G. P. Putnam's Sons 1961).