## Random Walk Algorithms: Lectures 13 \& 14

David A. Meyer

In which the discrete Fourier transform is derived, and used to diagonalize the transition probability matrix for homogeneous random walks in one dimension, and the problem of unblurring is revisited.

## The discrete Fourier transform

To diagonalize $X$, notice that $X^{N}=I$, so if $(\lambda, \vec{v})$ is an (eigenvalue, eigenvector) pair for $X$, i.e., $X \vec{v}=\lambda \vec{v}$, with $\vec{v} \neq 0$, then

$$
\vec{v}=I \vec{v}=X^{N} \vec{v}=\lambda^{N} \vec{v}
$$

so we can conclude $\lambda^{N}=1$. That is, if we set $\omega=e^{2 \pi i / N}$, the set of eigenvalues of $X$ is $\left\{\omega^{k} \mid k \in\{0, \ldots, N-1\}\right\}$.

To find the corresponding eigenvectors we must solve:

$$
0=\left(X-\omega^{k} I\right) \vec{v}=\left(\begin{array}{ccccc}
-\omega^{k} & & & & 1 \\
1 & -\omega^{k} & & & \\
& 1 & & & \\
& & \ddots & & \\
& & & 1 & -\omega^{k}
\end{array}\right)\left(\begin{array}{c}
v_{0} \\
v_{1} \\
v_{2} \\
\vdots \\
v_{N-1}
\end{array}\right)
$$

Setting $v_{0}=1$, this implies $1-\omega^{k} v_{1}=0$, so $v_{1}=\omega^{-k}$. Then $\omega^{-k}-\omega^{k} v_{2}=0$, so $v_{2}=\omega^{-2 k}$, etc. Normalizing the eigenvectors to have norm 1 gives

$$
\hat{f}_{k}=\frac{1}{\sqrt{N}}\left(\begin{array}{c}
1 \\
\omega^{-k} \\
\omega^{-2 k} \\
\vdots \\
\omega^{-(N-1) k}
\end{array}\right)
$$

Denote conjugate transpose by ${ }^{\dagger}$, and the inner product on $\mathbb{C}^{N}$ by $\langle u \mid v\rangle=u^{\dagger} v$. Then we can compute:

$$
\left\langle\hat{f}_{j} \mid \hat{f}_{k}\right\rangle=\hat{f}_{j}^{\dagger} \hat{f}_{k}=\frac{1}{N} \sum_{n=0}^{N-1} \omega^{n j} \omega^{-n k}=\frac{1}{N} \sum_{n=0}^{N-1} \omega^{n(j-k)}= \begin{cases}\frac{1}{N} \frac{1-\omega^{N(j-k)}}{1-\omega^{j-k}}=0 & \text { if } j \neq k \\ 1 & \text { if } j=k\end{cases}
$$

to conclude that the set $\left\{\hat{f}_{k}\right\}$ is orthonormal. Since the $\hat{f}_{k}$ are the columns of the diagonalizing matrix,

$$
F=\frac{1}{\sqrt{N}}\left(\begin{array}{ccccc}
1 & 1 & 1 & & 1 \\
1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\
1 & \omega^{-2} & \omega^{-4} & & \\
& \vdots & & \ddots & \\
1 & \omega^{-(N-1)} & & & \omega^{-(N-1)^{2}}
\end{array}\right)
$$

and $F^{\dagger} F=I=F F^{\dagger}$ by the orthonormality of the set $\left\{\hat{f}_{k}\right\}$, so $F^{-1}=F^{\dagger}$, i.e., $F$ is unitary. $F$ is called the discrete Fourier transform, perhaps first written down in this form by Sylvester [1].

## Diagonalizing the transition probability matrix

As we noted in Lecture 12, since $F$ diagonalizes $X$, it also diagonalizes $B$ :

$$
\begin{aligned}
& F^{-1} B F \\
& \quad=\frac{1}{4} F^{-1}\left(X+2 I+X^{-1}\right) F \\
& \quad=\frac{1}{4}\left(\left(\begin{array}{llll}
1 & \omega^{-1} & & \\
& & \ddots & \\
& & & \omega^{-(N-1)}
\end{array}\right)+\left(\begin{array}{llll}
2 & & & \\
& 2 & & \\
& & \ddots & \\
& & & 2
\end{array}\right)+\left(\begin{array}{llll}
1 & & & \\
& \omega & & \\
& & \ddots & \\
& & & \omega^{N-1}
\end{array}\right)\right)
\end{aligned}
$$

so the set of eigenvalues of $B$ is

$$
\left\{\left.\lambda_{k}=\frac{1}{4}\left(\omega^{k}+\omega^{-k}+2\right)=\frac{1}{2}(\cos k \theta+1) \right\rvert\, k \in\{0, \ldots,\lfloor N / 2\rfloor\}\right\}
$$

where $\theta=2 \pi / N$. Notice immediately that $\lambda_{0}=1$, with eigenspace spanned by $\hat{f}_{0}=$ $(1,1, \ldots, 1) / \sqrt{N} ; \lambda_{k}=\lambda_{N-k}$ for $0<k<N / 2$, with eigenspace spanned by $\hat{f}_{k}$ and $\hat{f}_{N-k}$; $\lambda_{N / 2}=0$, if $N$ even, with eigenspace spanned by $\hat{f}_{N / 2}=(1,-1, \ldots, 1,-1) / \sqrt{N}$; and also $\left|\lambda_{k}\right|<1$ for $k>0 . \hat{f}_{0}$ and $\hat{f}_{N / 2}$ are shown below: the black dots represent the elements of $\mathbb{Z} / N \mathbb{Z}$; the red dots above or below them the corresponding component of the eigenvectors; and with the red curves anticipating the next paragraph.


To get real eigenvectors for the two dimensional $\lambda_{k}$ eigenspace when $0<k<N / 2$, set

$$
\hat{c}_{k}=\frac{\hat{f}_{N-k}+\hat{f}_{k}}{\sqrt{2}}=\sqrt{\frac{2}{N}}\left(\begin{array}{c}
1 \\
\cos \theta \\
\cos 2 \theta \\
\vdots \\
\cos (N-1) \theta
\end{array}\right) \text { and } \hat{s}_{k}=\frac{\hat{f}_{N-k}-\hat{f}_{k}}{i \sqrt{2}}=\sqrt{\frac{2}{N}}\left(\begin{array}{c}
0 \\
\sin \theta \\
\sin 2 \theta \\
\vdots \\
\sin (N-1) \theta
\end{array}\right)
$$

and define $\hat{c}_{0}=\hat{f}_{0}$ and, when $N$ is even, $\hat{c}_{N / 2}=\hat{f}_{N / 2}$. It follows immediately from the orthonormality of the elements in the basis $\left\{\hat{f}_{k}\right\}$ that the the elements in $\left\{\hat{c}_{k}\right\} \cup\left\{\hat{s}_{k}\right\}$ also form an orthonormal basis. The diagrams below show $\hat{c}_{1}$ and $\hat{s}_{1}$, and then $\hat{c}_{2}$ and $\hat{s}_{2}$.


## Evolving eigencomponents

Since the sets $\left\{\hat{f}_{k}\right\}$ and $\left\{\hat{c}_{k}\right\} \cup\left\{\hat{s}_{k}\right\}$ are orthonormal bases, we can expand an initial probability distribution vector as a linear combination of elements in either:

$$
\vec{u}_{0}=\sum_{k=0}^{N-1}\left\langle\hat{f}_{k} \mid \vec{u}_{0}\right\rangle \hat{f}_{k}=\sum_{k=0}^{N / 2}\left\langle\hat{c}_{k} \mid \vec{u}_{0}\right\rangle \hat{c}_{k}+\sum_{k=1}^{\lceil N / 2-1\rceil}\left\langle\hat{s}_{k} \mid \vec{u}_{0}\right\rangle \hat{s}_{k} .
$$

The advantage of doing this is that we can compute $\vec{u}_{t}$ easily:

$$
\begin{align*}
\vec{u}_{t} & =B^{t} \vec{u}_{0} \\
& =\sum_{k=0}^{N / 2}\left\langle\hat{c}_{k} \mid \vec{u}_{0}\right\rangle B^{t} \hat{c}_{k}+\sum_{k=1}^{\lceil N / 2-1\rceil}\left\langle\hat{s}_{k} \mid \vec{u}_{0}\right\rangle B^{t} \hat{s}_{k} \\
& =\sum_{k=0}^{N / 2}\left\langle\hat{c}_{k} \mid \vec{u}_{0}\right\rangle \lambda_{k}^{t} \hat{c}_{k}+\sum_{k=1}^{\lceil N / 2-1\rceil}\left\langle\hat{s}_{k} \mid \vec{u}_{0}\right\rangle \lambda_{k}^{t} \hat{s}_{k} \tag{1}
\end{align*}
$$

from which we can conclude that

$$
\lim _{t \rightarrow \infty} \vec{u}_{t}=\left\langle\hat{c}_{0} \mid \vec{u}_{0}\right\rangle \hat{c}_{0}=\frac{1}{\sqrt{N}} \hat{c}_{0}=\frac{1}{N}(1, \ldots, 1),
$$

since $\left|\lambda_{k}\right|<1$ for $k>0$. Also, as $t \rightarrow \infty, u_{t, s}=1 / N+O\left(\left|\lambda_{1}\right|^{t}\right)$, since $\lambda_{1}$ has the largest norm of the $k>0$ eigenvalues.

## Unblurring

Returning to the problem of unblurring which motivated our diagonalization of $B$, we can now see that when $N$ is even, the null space is one dimensional, spanned by $\hat{c}_{N / 2}$, while when $N$ is odd, there is no null space. In fact, equation (1) suggests a tactic for unblurring: Given a blurred distribution $\vec{u}$, if we know $t$, we can compute

$$
\vec{u}_{0}^{\prime}=\sum_{k=0}^{\lceil N / 2-1\rceil}\left\langle\hat{c}_{k} \mid \vec{u}\right\rangle \lambda_{k}^{-t} \hat{c}_{k}+\sum_{k=1}^{\lceil N / 2-1\rceil}\left\langle\hat{s}_{k} \mid \vec{u}\right\rangle \lambda_{k}^{-t} \hat{s}_{k},
$$

which will be the same as the $\vec{u}_{0}$ from whence it came, except for any component $\vec{u}_{0}$ may have had in the $\hat{c}_{N / 2}$ subspace, which was completely erased by $B$.

This tactic not only fails to recover the $\hat{c}_{N / 2}$ component of $\vec{u}_{0}$, but suffers from other flaws: First, for large $t$, numerical imprecision will make it practically impossible to recover also the components with small-but-not-quite-0 eigenvalues. Second, in practice one might not know $t$. Developing unblurring algorithms based on more ideas than this simple tactic has been an active area of research $[2,3,4]$.

## References

[1] J. J. Sylvester, "Thoughts on inverse orthogonal matrices, simultaneous sign-successions, and tessellated pavements in two or more colours, with applications to Newton's rule, ornamental tile-work, and the theory of numbers", The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Series 4, 34 (1867) 461-475.
[2] A. S. Carasso, "Linear and nonlinear image deblurring: A documented study", SIAM Journal on Numerical Analysis 36 (1999) 1659-1689.
[3] P. C. Hansen, J. G. Nagy and D. P. O'Leary, Deblurring Images: Matrices, Spectra, and Filtering (Philadelphia, PA: SIAM 2006).
[4] T. Pouli, D. W. Cunningham and E. Reinhard, "A survey of image statistics relevant to computer graphics", Computer Graphics Forum 30 (2011) 1761-1788.

