10 April 2019

Random Walk Algorithms: Lecture 5

David A. Meyer

In which Markov processes are defined and random walks introduced as examples thereof.

Random walks on $\mathbb{Z}$

The random variables we defined in the last lecture to be the sum of a sequence of random "steps" are random walks.* More precisely, let $\Omega = \bigl\{f_1 f_2 \ldots \mid f_i \in \{\mathrm{H},\mathrm{T}\}, i \in \mathbb{N}\bigr\}$, and for $i \in \mathbb{N}$, define

$$S_i(\omega) = \begin{cases} +1 & \text{if } f_i = \mathrm{H}; \\ -1 & \text{if } f_i = \mathrm{T}, \end{cases}$$

where the events $\{f_i = \mathrm{H}\}$ are mutually independent. Let $X_t = \sum_{i=1}^{t} S_i$. Then a *random walk* on $\mathbb{Z}$ is the sequence of random variables $\{X_t \mid t \in \mathbb{N}\}$. This is an example of a *random process* (*stochastic process*), namely a set of random variables indexed by some set.

Definition. A *Markov process* is a stochastic process $\{X_t \mid t \in \mathbb{N}\}$ such that

$$\Pr(X_{t+1} = x_{t+1} \mid X_t = x_t, \ldots, X_1 = x_1) = \Pr(X_{t+1} = x_{t+1} \mid X_t = x_t),$$

i.e., the future depends only on the present, not on how we arrived at the present [4].

Theorem. *The random walk on $\mathbb{Z}$ is a Markov process.*

Proof. Notice that from the definition, $X_{t+1} = X_t + S_{t+1}$. Thus

$$\begin{aligned}
&\Pr(X_{t+1} = x_{t+1} \mid X_t = x_t, \ldots, X_1 = x_1) \\
&\quad = \Pr(X_t + S_{t+1} = x_{t+1} \mid X_t = x_t, \ldots, X_1 = x_1) \\
&\quad = \Pr(S_{t+1} = x_{t+1} - x_t \mid S_t = x_t - x_{t-1}, \ldots, S_2 = x_2 - x_1, S_1 = x_1) \\
&\quad = \Pr(S_{t+1} = x_{t+1} - x_t) \qquad \text{by the independence of the } S_i \\
&\quad = \Pr(X_t + S_{t+1} = x_{t+1} \mid X_t = x_t) \\
&\quad = \Pr(X_{t+1} = x_{t+1} \mid X_t = x_t). \qquad \blacksquare
\end{aligned}$$

* This name was first used by Karl Pearson on July 27, 1905, when he asked for the probability density function for the distance from the starting point after n steps in random directions in two dimensions [1]. Lord Rayleigh pointed out two days later [2] that he had answered the question 25 years earlier [3].

Transition probabilities

These conditional probabilities are those we have already seen:

$$\Pr(X_{t+1} = x_t + 1 \mid X_t = x_t) = p$$
$$\Pr(X_{t+1} = x_t - 1 \mid X_t = x_t) = 1 - p.$$

More generally:

DEFINITION. Let S be a countable set and let $\{X_t \mid t \in \mathbb{N}\}$ be a Markov process with $X_t : \Omega \to S$, the set of *states*. The conditional probabilities

$$p_{ij}(t) = \Pr(X_{t+1} = i \mid X_t = j)$$

are called the *transition probabilities*, and the arrays $P(t) = \{p_{ij}(t) \mid i, j \in S\}$ are called the *transition matrices*. Notice that $\sum_{i \in S} p_{ij}(t) = 1$, for all $j \in S$; this, together with the nonnegativity of its entries, makes $P(t)$ a *stochastic* matrix. We will mostly consider cases in which the transition matrix does not depend on t; these are *homogeneous* Markov processes.

For a random walk on $\mathbb{Z}$, the transition probability matrix is

$$P = \begin{array}{c} \\ \vdots \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} \ldots & -2 & -1 & 0 & 1 & 2 & \ldots \end{array} \\ \begin{pmatrix} & & & & & & \\ & \ddots & 1-p & & & & \\ & & 0 & 1-p & & & \\ & & p & 0 & 1-p & & \\ & & & p & 0 & & \\ & & & & p & \ddots & \\ & & & & & & \end{pmatrix} \end{array}.$$

To define a random walk with a *finite* state space, consider the quotient space $\mathbb{Z}/\ell\mathbb{Z} = \{[x] \mid x \in \mathbb{Z}\}$, where the equivalence class $[x] = \{x + \ell\mathbb{Z}\} = \{x + k\ell \mid k \in \mathbb{Z}\}$. This is the integers modulo ℓ, and a random walk on them can be defined with the same rule: $X_{t+1} \equiv X_t + S_{t+1} \pmod{\ell}$. Then the transition probability matrix is

$$P = \begin{array}{c} \\ [0] \\ [1] \\ \vdots \\ [\ell-2] \\ [\ell-1] \end{array} \begin{array}{c} \begin{array}{ccccc} [0] & [1] & \ldots & [\ell-2] & [\ell-1] \end{array} \\ \begin{pmatrix} 0 & 1-p & & & p \\ p & 0 & & & \\ & \ddots & \ddots & \ddots & \\ & & & 0 & 1-p \\ 1-p & & & p & 0 \end{pmatrix} \end{array}.$$

References

[1] K. Pearson, "The problem of the random walk", *Nature* **72** (1905) 294.

[2] Lord Rayleigh, "The problem of the random walk", *Nature* **72** (1905) 318.

[3] Lord Rayleigh, "On the resultant of a large number of vibrations of the same pitch and of arbitrary phase", *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, Series 5 **10** (1880) 73–78.

[4] A. A. Markov, "Extension of the limit theorems of probability theory to a sum of variables connected in a chain" (1906); translated and reprinted in R. A. Howard, *Dynamic Probabilistic Systems, Volume 1: Markov Chains* (New York: John Wiley and Sons 1971) Appendix B.